

## IN THE TOWER OF BABEL: BEYOND SYMMETRY IN ISLAMIC DESIGN

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**Abstract**—A personal account of an interdisciplinary inquiry into the study of Islamic geometric design and architectural decoration touching on the fields of History, History of Science, Scientific Theory of Symmetry and History of Art. The study stresses the necessity of the use of a common scientific language of Symmetry Notation in order to discuss and communicate in a precise manner about Islamic geometric pattern. To understand Islamic geometric design, it is necessary to move beyond the symmetry issues, to the step-by-step process of design. This is based on primary sources of scientific manuscripts of practical geometry written specifically for the Muslim artisans. The research demonstrates not only a direct meeting but a collaborative work between science and art in Islamic civilization.

The story of arrogant men building the Tower of Babel (Genesis 11) reads as follows:

"Now the whole earth had one language and few words. And as men migrated from the east, they found a plain in the land of Shinār and settled there. And they said to one another, 'Come, let us make bricks, and burn them thoroughly.' And they had brick for stone, and bitumen for mortar. Then they said, 'Come, let us build ourselves a city, and a tower with its top in the heavens, and let us make a name for ourselves, lest we be scattered abroad upon the face of the whole earth.' And the Lord came down to see the city and the tower, which the sons of men had built. And the Lord said, 'Behold, they are one people, and they have all one language; and this is only the beginning of what they will do; and nothing that they propose to do will now be impossible for them. Come, let us go down, and there confuse their language, that they may not understand one another's speech.' So the Lord scattered them abroad from there over the face of all the earth, and they left off building the city. Therefore its name was called Bābel, because there the Lord confused the language of all the earth; and from there the Lord scattered them abroad over the face of all the earth."

This paper examines the relationship between the fields of "History, History of Science, Scientific Theory and the Process of Islamic Geometric Design". The theme of the Tower of Babel and the curse of the multiplicity of tongues runs throughout my discussion of these topics because of the lack of a common language, either in the study of Islamic geometric pattern or in interdisciplinary discourse. If the curse of Babel has beset us in the field of Islamic Art, we need not despair, for there is hope that it can be rectified. In the Biblical analogy, the curse from the Old Testament is finally removed in the New Testament. It is only through the grace of God and the true demonstration of love by man that the curse of multiplicity of language with the resulting confusion of tongues is to be undone and people will be able to understand each other, as on the day of Pentecost (Acts 2:7).

My approach is frankly etiological. Just as the etiological passages from the Tower of Babel in the Old Testament explain how things came to be in the world (Fig. 1), this paper explains how things came to be in my attempt to study and document the direct meeting of science and art in Islamic civilization. It also takes up the background and origins of the methodological and interpretive trends that have characterized the study of the history of Islamic art and Islamic geometric pattern and ornament. I have been studying this material for 17 years, beginning in 1970. This account will span the first 7 years of my research in this area, from 1970 to 1977. The material is presented chronologically, and in a somewhat personal manner.

I begin with the process through which I originally came upon the material because of the crucial importance of tools and methodology to this kind of work. Information of this kind is rarely revealed publicly, nor is it often published. It is from these starting points, too often ignored, that one can learn most, for they involve more than tools and methodology; they involve the logical process of interdisciplinary research.

My account begins in 1971, with a meeting of a small group of people interested in Islamic art and architectural decoration. We were all looking at the same monument and the same portion of its decoration. In each case, our descriptions, analyses and even the naming of parts and shapes of



Fig. 1. The Tower of Babel: the Confusion of Tongues. Illustrated by Gustave Doré in *The Doré Bible Illustration* (Dover, New York).

that decoration were totally different from those of the person sitting next to us. We all saw what we saw, and we each spoke in our own terms and in our own language. We walked out as if we had not been together, we had not communicated, we had not understood each other. We were all speaking different languages. From that very early moment, I realized that there was something amiss in the study of Islamic architectural decoration. We lacked the proper tools, and proper or common language. We would never be able to truly classify, analyze and comprehend the material if we continue in this way.

Shortly thereafter, on a memorable afternoon, as I skimmed through books on Islamic architecture, I made the simple observation that, since the tenth century, an increasing number of geometric

figures were used with a parallel increase in sophistication of the patterns of geometric design. These observations led to an obvious question: whoever had created these elaborate geometric designs must have mastered a knowledge of practical geometry that enabled him to have achieved the resulting structures or geometric patterns. If the Muslim artists, artisans, architects, builders, designers, carpenters and craftsmen knew geometry, they could not have acquired it spontaneously. They must have learned it, and therefore they must have been taught. But how were they taught? What knowledge of geometry was available for teaching? Who was teaching, and with what books or manuals? If such textbooks or manuscripts existed, then we should look for them, study their nature, clarify the problems that they resolved, distinguish what they considered as problematic in their own materials and find the geometric methods of construction they used to achieve the designs and patterns that are now recognized as artistic masterpieces. Such an approach would bring us closer both to an objective comprehension of the methods of design these artisans used and to understand the step-by-step process of Islamic geometric design.

In the summer of 1971, as I started my search for a thesis topic, the geometry and architectural decoration was still on my mind. I cannot help recall the initiation of this now rewarding project. I explained my observations about the development of Islamic geometric designs to my advisor and I expressed my wish to find a textbook of geometry or manuscript that was written specifically for artisans to teach them how to design and to study the manuscript in order to come to an objective understanding of Islamic geometric design and architectural decoration. The immediate response was that there was no such thing. To this I responded that I would look for it, and only if I don't find it would I be able to say there is no such thing. Thus, I was faced with a most emphatic assertion that my proposal and preliminary conclusion would never be confirmed by finding the material evidence of manuscripts written for the artisans, and the whole project was doomed to failure.

Such quick and categorical negation was typical of a widespread assumption on the part of those in the field that no such manuscripts or written documentation ever existed, an assumption that proved to be invalid. I was asked to broaden this topic in anticipation of failure at my original questions. I therefore included other relevant issues, such as a survey of those aspects of Islamic civilization that seem to reflect the widespread interest in geometry. This was to guard against failure, and possibly to serve as a stepping stone for documenting the influence of geometry in both the arts and society in general. The list of questions began to expand as I set about the task of finding the artisans' textbooks. For example, can one show that the interest in science or geometry was part of the average cultured person's background in the ninth or tenth century? What practical geometry had been developed by the tenth century? What caused the growth of this phenomenon? Geographically, where did it begin and in what directions did it spread?

My ultimate aim remained, however, to find out what kind of geometry was taught to artisans; what they knew; what problems they faced in their design; and, if geometric theory was available to artisans, how long it took before it was no longer the exclusive property of the scientific community. When did it filter down to the craftsmen and architects? Did science and art not only meet, but actively collaborate in Islamic civilization?

I tackled these questions from the standpoint of a strong background in art and in Islamic studies and with a general background in history, historiography and research methodology. The last three were due to two outstanding professors, Constantine Zuraik of the American University of Beirut and George Maqdisi, then at Harvard. Their training provided the tools that are necessary to take advantage of the extensive resources of the Harvard library system. I read through catalogues and indices of manuscript collections available in libraries throughout the world. By the end of the week I had a large pile of index cards referring to geometry manuscripts. I sorted them to see which were edited, which had known authors, which had known contents, which were in what library or city, etc. And as I was going through the cities card stack, it struck me that there were several manuscripts in the Khudabakhsh Library in Patna, India. Another look at the index cards revealed that a number of the Patna manuscripts were copied in the city of Mosul in northern Mesopotamia, very early in the thirteenth century (632 A.H./1234 A.D.). Two questions came to mind simultaneously: (1) How did those manuscripts get to Patna? (2) Someone must have been studying or working on geometry in Mosul *ca* 632 A.H./1234 A.D. Who was that?

Research on the history of Muslim schools and teaching in the thirteenth century revealed that there were two very famous scholars in Mosul at the time: Kamāl al-Dīn Yūnis bin Man'a and Athīr

al-Dīn al-Abharī. The former was recognized as the most outstanding teacher at the Muslim main school of the time in Mosul (the school was subsequently named after him “al-Madrasah al-Kamālīyah”).

I reported the results of this first week of research—how I was led to Mosul and the teaching of Kamāl al-Dīn Yūnis bin Man‘a whose tracks and findings I proposed to follow. To my surprise, this plan was immediately dismissed on the grounds that what I had identified was due to the Nestorian Church and activities related to its revival. I was dismayed, especially as the name of the teacher in Mosul seemed very Muslim. I continued my search for information about Kamāl al-Dīn Yūnis bin Man‘a, and immediately discovered that an incredible amount of information was available. Nearly every historical source of that period that I consulted had an entry on him, and these included: Ibn Khallikān’s history, *Obituaries of the Eminent Men and Histories of the Leading Contemporaries*; Ibn abī Uṣaybi‘a’s history, *The Choicest News of the Generations of Physicians and Scholars*; Ibn al-Fūwaṭī’s work on thirteenth century history, *Comprehensive Occurrences and Beneficial Experiences of the Seventh Century of Hijra*; and, most incredibly, the history of the Muslim scholars of the *Shāfi‘ī* theological school, *The Great Classes of Shāfi‘ī Scholars* of al-Subkī [1]. This last source clearly classified Ibn Man‘a among the outstanding Muslim scholars of the *Shāfi‘ī* school. Brockelmann’s *History of Arabic Literature* [2] listed his writings and revealed that there was at least one manuscript available of his works that could be of great interest to this research. To my amazement, a transliterated title appeared on Brockelmann’s pages: *Risāla fīmā yaḥtāju ilayhī al-ṣāni‘u min a‘māl al-handasa*, on which Kamāl al-Dīn Yūnis bin Man‘a had written a commentary entitled *Sharḥ al-a‘māl al-handasiyya*. The title of the main work, or the subject of Ibn Man‘a’s commentary, means literally “A treatise on what the artisan needs of geometric problems” while the title of the commentary is “Commentary on the geometric problems”. This main title corresponded exactly to the contents of the manuscript geometry textbooks I had visualized, yet I had never dreamt that it would be the actual title of a manuscript.

The work turned out to be that of Abū‘l-Wafā’ al-Būzjānī, a well-known scientist and mathematician who lived in Baghdad from approximately 945 A.D. until his death about 987 A.D. The geometry text was singled out as being an important document and of specific interest to the historians of Islamic art in 1855 by the Austrian historian of science, F. Woepcke, and there were a number of recensions of the manuscript available in various libraries across the world [3]. Thus by the third week I had located an example of the kind of manuscript I was looking for and identified its author.

At this point, several observations are in order. First, simple but correct reasoning and logic are the basis for most research. In general one should not deny the existence of something without first looking for it. Second, the emphatic response to my topic proposal, “there is no such thing”, is quite revealing, and seems to reflect the prejudices of Western scholars after World War I. To them, Islamic civilization could not have been intellectual. They assumed that Muslim craftsmen were people of minimal knowledge and education, capable only of minimal creative expression, whose genius, if they were uncommonly intelligent, consisted exclusively in committing two or three patterns to memory. Their lifetimes were spent merely reproducing those two or three patterns. Such opinions also proliferated as a result of anecdotal stories told by earlier English travelers, for example, the following quotation from Archibald Christi:

“Oriental workers carry intricate patterns in their heads and reproduce them easily without notes or guides. There is a story that tells of an English observer, seeing a most elaborate design painted directly on a ceiling by a young craftsman, [the observer] sought the artist’s father to congratulate him on his son’s ability, but the father replied that he regarded the boy as a dolt for he knew only one pattern, but his brother was indeed a genius—he knew three!” [4]

Ignorant Islamic craftsmen supposedly knew only the number of their ten fingers and this indicated the limit of their intelligence or education. So imbedded was this ridiculous assumption that in the summer of 1971 inquiry into the craftsmen’s knowledge of geometry seemed absurd. No one inquired as to who designed those sophisticated patterns and how.

The third point is the dismissal of the possibility that observable scientific activity in the thirteenth century was Islamic, and its attribution to the Nestorian Eastern Church or its revival, i.e. to Christian civilization. The assumption that Islam and Islamic civilization brought nothing new, that the Umayyad period simply inherited from the Byzantine and merely reproduced it in distorted

ways, is an overwhelming barrier to progress in the field. Islamic civilization was never given the slightest chance of being scientific or pragmatic and of having intellectual activities. Thus, Kamāl al-Dīn Yūnis bin Man‘a was dismissed as a Christian from the Nestorian Church. Scientific geometry textbooks written for the artisans/architects were non-existent.

Three weeks after beginning my search, I located the artisans's geometry textbooks. Before departing for Europe, I found at the Harvard Fogg Library a reference in a book on ornament from 1910 to 1911, indicating the presence of a collection of architect's drawings, the "Mirza Akbar Collection" at the Victoria and Albert Museum in London [5]. There, it took the staff five days to locate the collection in storage. I was astounded by the number of drawings it contained and by the size of the collection. The staff of that section shared my surprise and gathered around the table in amazement at these drawings, which are of an architect's workshop from the eighteenth and nineteenth centuries. In 1981, I examined similar material in two Arab towns; it is still in the hands of the artisans today. These scrolls (Fig. 2) were not only the basic reference manual, but also served as a design book from which artisans chose the appropriate pattern to be used in architectural decoration or in the workshop.

My next stop was Paris to examine a Persian translation of the manuscript of Abū al-Wafā, al-Būzjānī. I had taken with me a shopping list, as George Maqdisi used to call it, of other items of possible interest in that library. The shopping list for the Bibliotheque Nationale included a manuscript without title or author, mentioned in the catalog only as "a manuscript of geometry problems with geometric figures" [6]. My first glance at the folios of this manuscript made it clear that this was a much more important finding than the manuscript of Abū al-Wafā'. Here, the complex geometric patterns of design were recognizable in the drawings of the repeat units, which were distinctively illustrated. Also, in contrast to the simple shapes and polygons of Abū al-Wafā's manuscript, the complex geometric shapes in this manuscript, "On interlocking similar and congruent figures", indicated a much higher and later stage of development.

By the time I returned to Cambridge, I had located a range of written material, in the history of Islamic science and geometric design from the tenth century to the mid-nineteenth century, lying in library and museum storage rooms all over the world. In point of fact, my material turned out to be so convincing that it is now being used and propagated even by those who demonstrated such a strong skeptical attitude towards it at the beginning. Though locating the manuscripts took only two months, acquiring microfilms and/or photocopies of these documents without any backing or support took several years. Meanwhile, I was struggling to decipher the material, and to find an appropriate language in which to discuss it and describe the geometrical patterns with which it dealt.

At a very early point in my research, I was aware of the existence of an appropriate scientific language, namely that of Group Theory and Crystallography. As early as 1944, Edith Müller had written a thesis on Group Theory and Symmetry Notation of the patterns of Moorish ornament in the al-Hambra Palace [7]. Earlier, Andreas Speiser had called particular attention to Islamic art in his chapter on ornament in *Die Theorie der Gruppen von endlicher Ordnung* as early as 1927 [8]. It was only in 1935 that the scientific findings of Point Group Theory were fixed in the International Tables of Crystallography. These notations became the most widely used language by the chemists. It is possible that E. Müller had followed the line of inquiry that A. Speiser had suggested. She deserves considerable credit for systematically articulating the connection on a monument. Her thesis work takes into account the scientific theoretical finding and reveals the value of these scientific theories to the understanding and classification of Islamic geometric patterns. This information was neglected by all historians of Islamic art until the early eighties, and no one has even attempted to bring up the issue within the field. It is possible that the difficulty of her scientific language makes her material and research inaccessible to the historian of art. As I tried to read that book, I realized that it was too complicated and too scientific for either students of Islamic art history or general art historians. Thus E. Müller's study remained the only one to apply Group Theory and Symmetry Notation to the study of Islamic geometric pattern.

To understand her book, I needed to understand scientific theory, and I spent endless days at the chemistry library studying the basics of Group Theory and the theory of symmetry and its notational systems. I found that the language was difficult and did not completely serve my purposes. There were so many different notation systems for the symmetry groups that it was confusing for an outsider to the field of chemistry to attempt to evaluate or select one. This was especially true

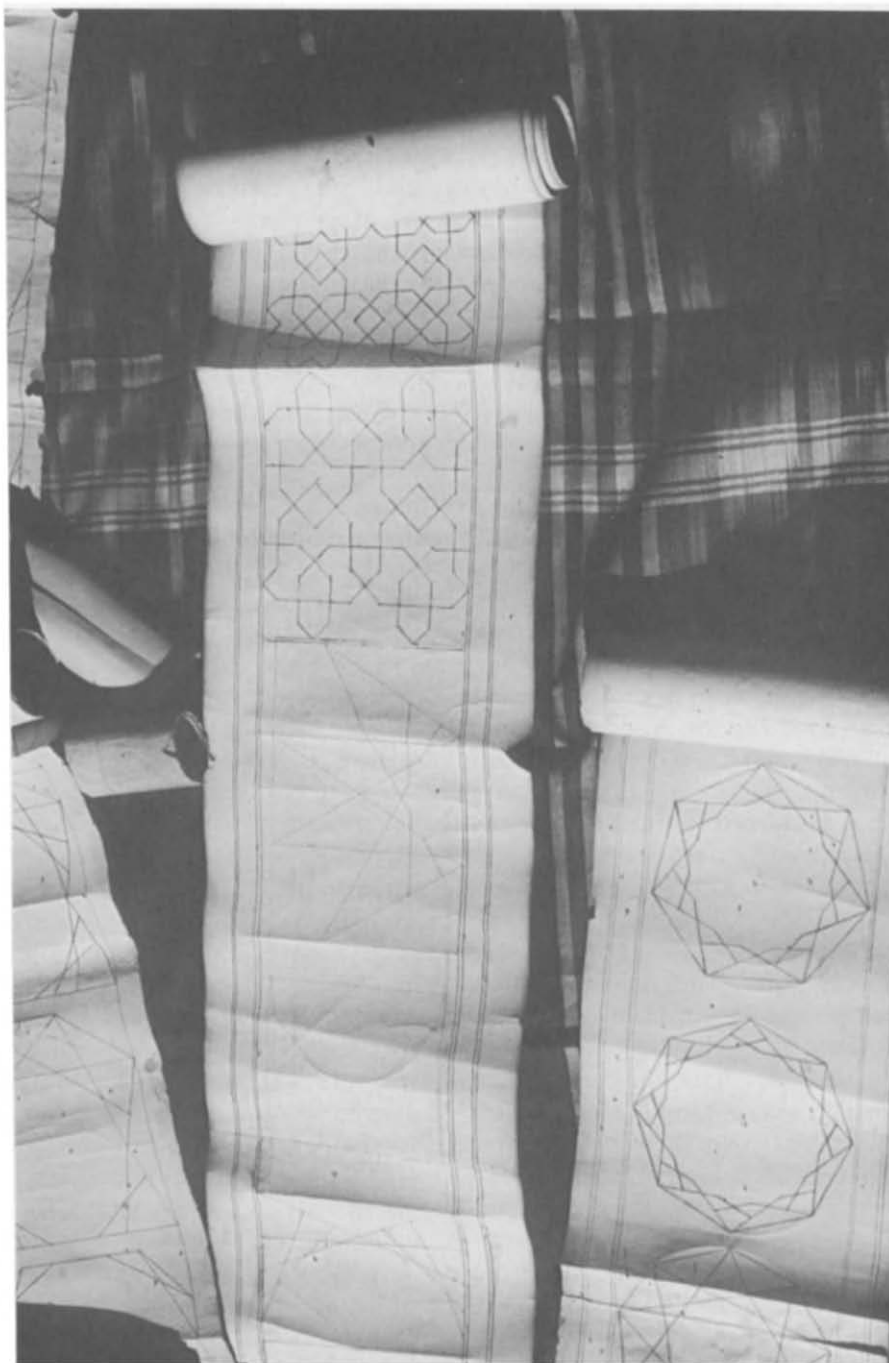


Fig. 2. paper scrolls with geometric patterns and drawings from the master-artisan's workshop in an Arab town 1982.

because I needed this notation as a tool to help classify the material that I was studying: I did not intend to add to the existing confusion in the field or Art History, specifically, the chaotic state of loss into which the study of pattern and ornament had fallen after an incredible blossoming at the end of the nineteenth century.

By the mid-seventies, the field of Islamic Art experienced an overwhelming surge of publications and revived interest. The number of books published increased tremendously relative to the previous few decades. Yet the standard of the publications was academically at its nadir, especially the study of geometry and ornament. This chaotic situation involves three concerns:

1. The preoccupation in the field of Islamic Art with issues of patronage—Royal, Princely and Mystical.

2. The proliferation of visual description and the psychology of visual perception as approaches to the studies.
3. The popularity of Linguistics, and later Semiotics, as the “sciences” within the arts whose languages were most developed, and that could be used as tools to attain a more scientific level of understanding of geometric pattern and art.

The shortcomings of the above concerns all point to the need for a scientific language and methodology with which to understand and systematically categorize Islamic geometric pattern.

### 1. Preoccupation in the Field of Islamic Art with Issues of Royal–Princely and Mystical Patronage

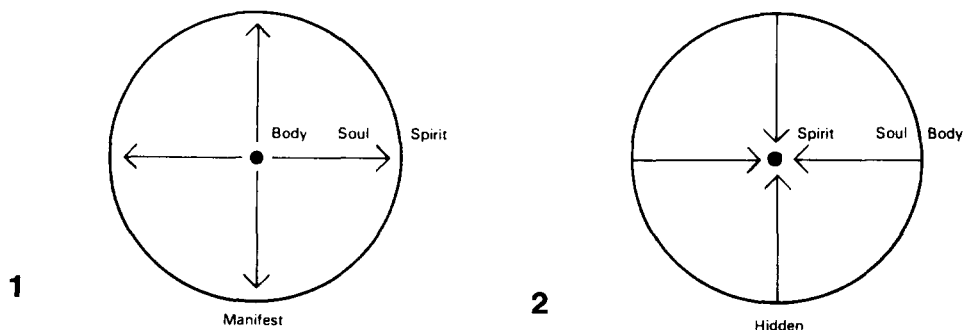
In the mid-seventies, there was a general preoccupation in much of the field with issues of patronage, both Royal–Princely and Mystical. I will not go into the wide range of problems created by the concentration on the Royal–Princely patronage or how and why it gained the main grounds in the field. Perhaps it was no coincidence that this was the period when oil money flowed and art historians attempted to lure these moneys to their field. In those years, the moneys of the imperial courts of Persia, in particular, played a very active role in the Islamic art scene. Exhibitions, conferences and publications multiplied.

A very well-known, active group of international mystics supported specific publications and pushed certain Islamic mystical ideas. Their main doctrine was that of the “Principle of the Unity of Being”. They tried to show that, ultimately, all differences in outward manifestations are “inwardly united at the Center. They are the bridge from the periphery to the Center, from the relative to the Absolute, from the finite to the infinite, from multiplicity to Unity”, quoting Seyyed Hossein Nasr [9]. The introductions and forewords of a number of books permeating such mystical themes were written by either Seyyed Hossein Nasr or Titus Burckhardt. These books include: *The Sense of Unity: the Sufi Tradition in Persian Architecture* by Nader Ardalan and Laleh Bakhtiar; *Sufi: Expression of the Mystic Quest* by Laleh Bakhtiar; *Islamic Patterns* by Keith Critchlow; and *Geometric Concepts in Islamic Art* by Issam el-Said and Ayse Parman. Frequently S. H. Nasr’s main ideas are illustrated in geometric drawings in these books or are used in long quotes. A typical example is the circle with its center (Fig. 3) from N. Ardalan’s *The Sense of Unity*, where these two drawings of the (*Zāhir*), the manifest of apparent, and the (*Bāṭin*), the hidden or internal, are represented as the center of the circle, stand for body and soul, respectively:

“The Manifest (*Zāhir*): The Consideration of God as Hidden and the Manifested pertain to ‘space’—to ‘qualified’ and ‘sacred’ space . . . Taken as Manifested, God becomes the reality that englobes all, that ‘covers’ and encompasses the cosmos. In this view, physical manifestation may be regarded as the innermost circle of a set of five concentric circles, followed by the other states of being respectively, with the outermost circle symbolizing the Divine Essence. . . .” [S. H. Nasr, *Science and Civilization in Islam*, p. 93.]

“The Hidden (*Bāṭin*): [This] can be regarded as symbol of the microcosm, of man, in whom the physical is the most outwardly manifested aspect and his spiritual nature the most hidden. . . .” [S. H. Nasr, *Science and Civilization in Islam*, p. 94.] [10]

The “Principle of the Unity of Being” pervades the content of these works, sometimes even their titles. On occasion, it is pushed to a point of scientific fallacy such as the claim that all geometric patterns of Islamic art are derivable through a single method of construction based on the subdivision of the circle, in order to declare this art work an example of the “Unity of Being”. This



Figs 3.1 and 3.2. Mystical symbolism applied to the circle and its center in N. Ardalan’s *The Sense of Unity* [10].

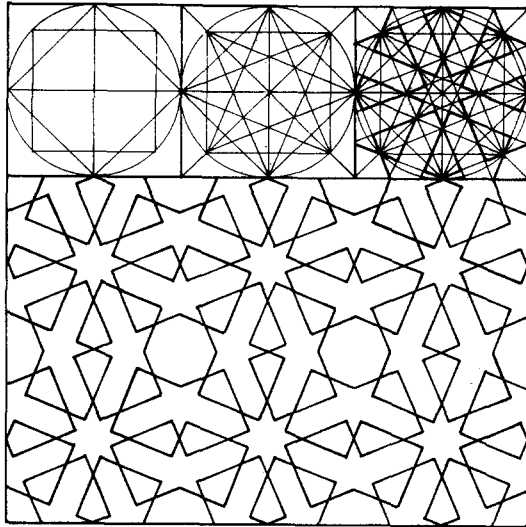


Fig. 4.1. The stress upon subdivision of the circle in deriving all geometric patterns. From I. El-Said, *Geometric Concepts in Islamic Art* [12].

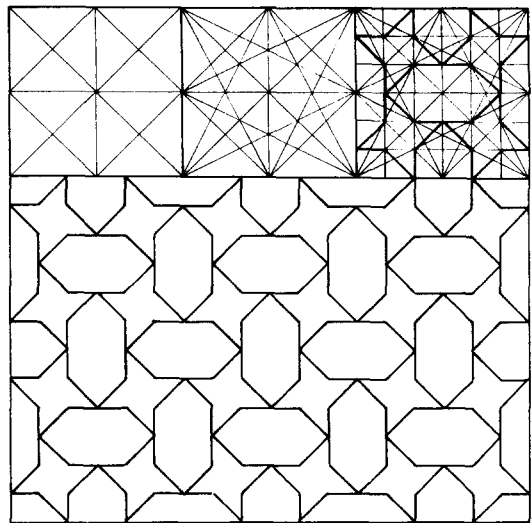


Fig. 4.2. The circle does not appear in deriving this geometric pattern. From I. El-Said, *Geometric Concepts in Islamic Art* [13].

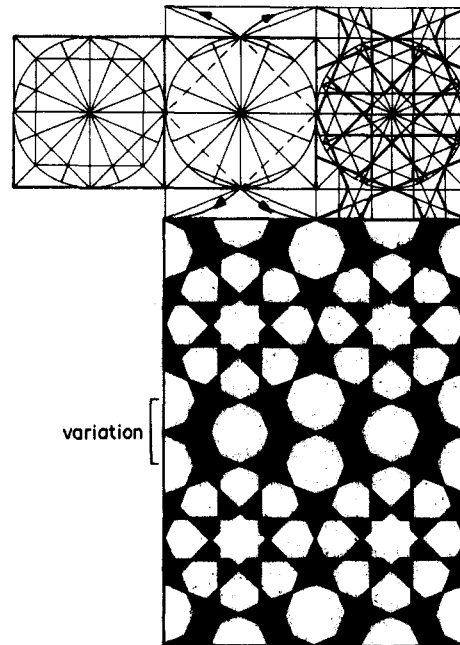


Fig. 4.3. The scheme of the circle does not fit the long rectangular unit, a variation zone is marked. From I. El-Said, *Geometric Concepts in Islamic Art* [14].

argument appears in *Geometric Concepts in Islamic Art* by I. El-Said. In his introduction to it, Titus Burckhardt states that all the geometric patterns are derived by the same “method of deriving all the vital proportions of a building (or a pattern) from the harmonious division of a circle . . . which is no more than a symbolic way of expressing Unity (*Tawhīd*), which is the metaphysical doctrine of Divine Unity as the source and culmination of all diversity” [11]. This is illustrated in the mesh of the subdivisions of the circle from which the pattern is developed (Fig. 4.1) [12]. In some cases, however, the authors neglected to draw in the circle, ironically revealing how unfundamental its existence is to the alleged “unique way” or “only way” of deriving all patterns (Fig. 4.2) [13]. And finally, there are a few cases of designs where it was absolutely impossible to hide the fact that the analytical method did not hold. These are illustrated (Fig. 4.3) as containing a non-standard zone



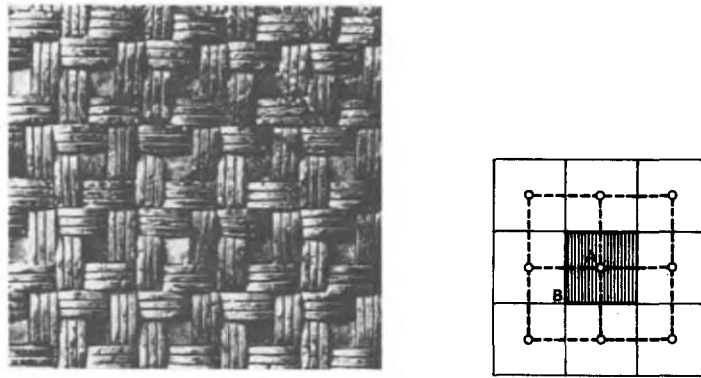


Fig. 5.1. Mystical interpretation given to the two distinct 4-fold symmetry centers of brickwork by L. Bakhtiar in *Sufi: Expressions of the Mystic Quest* [15].

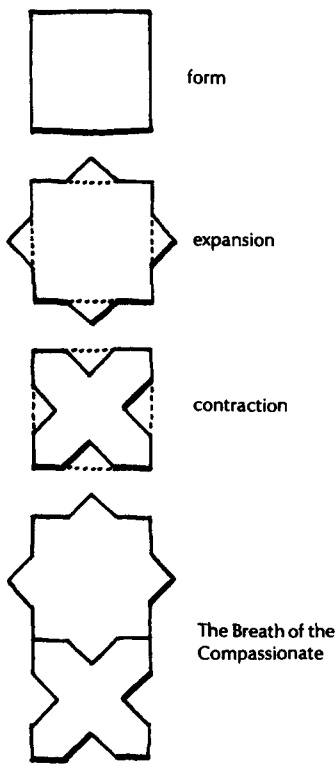


Fig. 5.2. Mystical symbolism applied to the 8-pointed star and cross pattern by L. Bakhtiar in *Sufi: Expressions of the Mystic Quest* [16].

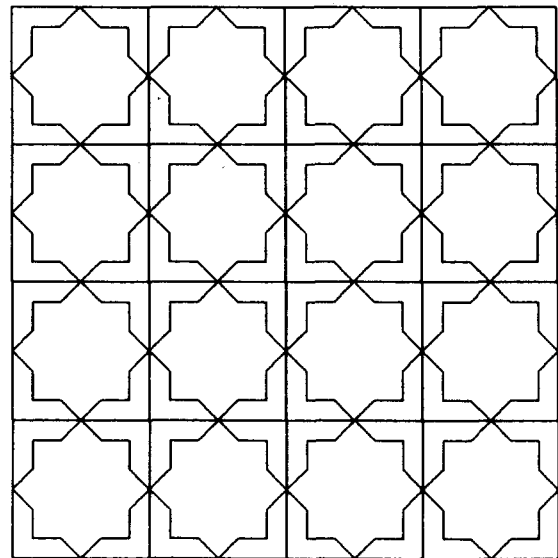


Fig. 5.3. Very popular ceramic pattern with an 8-pointed star and cross pattern.

labeled as a “variation”! [14] The elongated rectangular area obviously belongs to a 2-fold symmetry group and cannot be masked by the generalization of the “one way” that is overwhelmingly represented in the square of a 4-fold symmetry group.

We know from scientific theory that there are 17 different and distinct groups of two-dimensional patterns that are periodic in two independent directions. The laws of symmetry for these 17 patterns have been established and recognized by the international scientific community of crystallographers since the mid-thirties. Yet here we are in the mid-seventies not acknowledging this fact and declaring that there is only one way to draw all the patterns. One cannot help but point out that the theme of the book *Geometric Concepts in Islamic Art* involves a scientific fallacy, in order to meet the demands of the desired mystical interpretation.

As to the last book, L. Bakhtiar's *Sufi: Expressions of the Mystic Quest*, a few remarks will suffice. In the right-hand image of the brickwork (Fig. 5.1) the points A and B, which are the distinct roto-centers for a 4-fold rotational symmetry operation 4 and 4', are declared to be "shuhūd, presence [conscious witnessing of God's presence] (A), and ghabat, absence [unconscious witnessing of God's presence] (B), in a brickwork pattern at the Masjid-i-Hakīm Isfahān, Iran" [15]. The most popular pattern in Islamic geometric design, the interlocking 8-pointed star and cross (Fig. 5.2), becomes "form, expansion, contraction, the Breath of the Compassionate"! [16] This pattern exists in simple brickwork, in tiles, in wood and in solid gold! I wonder if the artisan who made this design thought of it as form, expansion, contraction and the Breath of the Compassionate God? Is this not a simple geometric design that involves a 4-point rotational symmetry? (Fig. 5.3) It is one thing to believe in mysticism and to follow in its practices and experience its positive effects. But it is a totally different matter when a new set of interpretations and symbols is created and propagated under the guise of historical truth. The symbolic mystical interpretations that have proliferated in these books on Islamic geometric design, pattern and ornament are based on a modern understanding of Islamic literature. There is no documented evidence that such interpretations were given to the art forms when they were created hundreds of years ago.

Earlier writings of S. H. Nasr reveal the core of a program to educate modern man to understand the language of symbolism in order to revitalize traditional sciences. He declares in *Man and Nature: the Spiritual Crisis of Modern Man* that:

"Such a revitalization of the traditional sciences, however, requires a re-discovery of the true meaning of symbolism and the education of modern man to understand the language of symbolism in the same way that he is taught to master the languages of logic or mathematics."

The general public unfortunately remains unaware of this. If in these books, that are now readily available on the market, their authors had made clear that the presented views were modern understandings of old forms, turning them into symbols, there would be no reason to object. The problem lies in presenting these modern mystical views as historical truths, as if these symbols were the meanings at the time the art forms were created. The non-Islamicist who is exposed to these books will anachronistically assume that a modern interpretation is the historical truth. Where does one draw the line between true historical research and the creation of and attribution of symbolic meaning to forms from the past? How can we redeem the geometric shapes, forms and patterns from the shrouds of mystical interpretations in order to see the precise scientific design at their basis?

### 2. Visual Description, Perception and Arnheim

Meanwhile, a sudden surge in the popularity of several approaches in the theoretical field of Art History overwhelmed Islamic art. This situation has created an urgent need to extract the field from superficial analyses that do not contribute to the understanding of geometry and pattern.

The most prominent of these approaches was that of visual perception and visual description, popularized by Rudolf Arnheim, whose book had just been published and who was teaching at Harvard [18]. This art-historical approach stressed the process of selective vision, and its main approach to the study and understanding of art was based on visual description and on the psychological interpretation of perception, including such elements as balance, movement and tension. The emphasis was on what one sees and the gradual mechanics of what and how one sees. Thus, looking at the graphic drawings of a brick wall from the tomb of Muḥammad Makkī al-Zangānī in Kharrāqān (486 A.H./1093 A.D.) through the detailed sequence in Figs 6.1–6.3 the viewer attempts to depict the possible selective mechanics of visual perception and description of the pattern. A person might first notice the V-forms, then the vertical X-forms or the horizontal X-forms, and only then realize that there are dots or circles, and need to balance them, connect them and see a square relationship or a bilateral relationship in the grouping. Similarly, the viewer starts perceiving a vertical or horizontal orientation, rhythm and repetition. As long as a person is looking, and whenever there is a stop or visual pause, there is an immediate process of selection and new perception. In all this, we remain on the surface level of description.

### 3. Linguistics and Semiotics as Fashion

While looking at these small geometric shapes (Figs 6.2 and 6.3) or similar ones, I often heard people proclaim loudly in discussions that this is a phoneme, this is a moneme or this is a signifier,

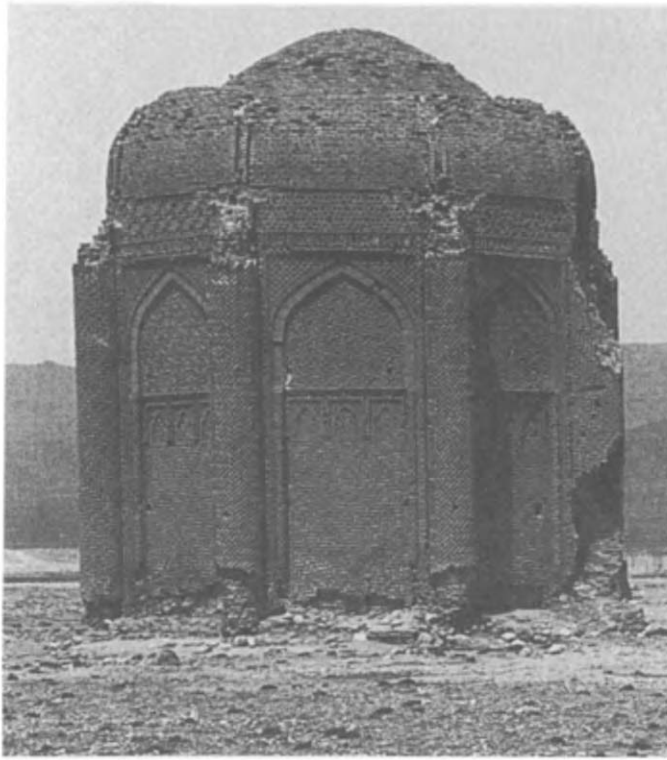


Fig. 6.1. General view of the Kharraqaq tomb tower 1093 A.D. Photo: S. P. and H. N. Seherr-Thoss, *Design and Color in Islamic Architecture*, p. 53. Smithsonian Institution, Washington, D.C. (1968).

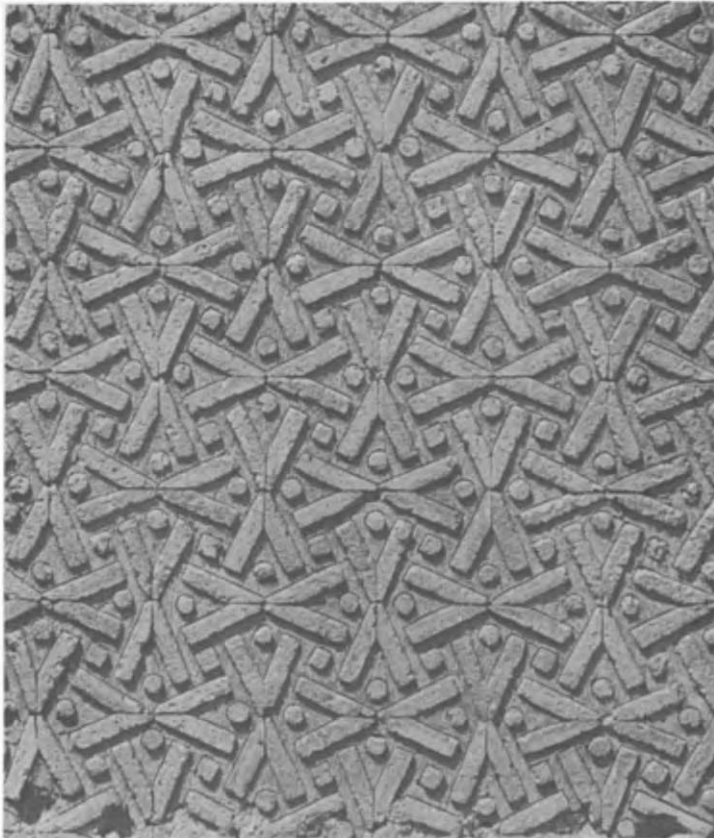


Fig. 6.2. Brickwork geometric pattern from the Kharraqaq tomb tower 1093 A.D. Photo: S. P. and H. N. Seherr-Thoss, *Design and Color in Islamic Architecture*, p. 61. Smithsonian Institution, Washington, D.C. (1968).

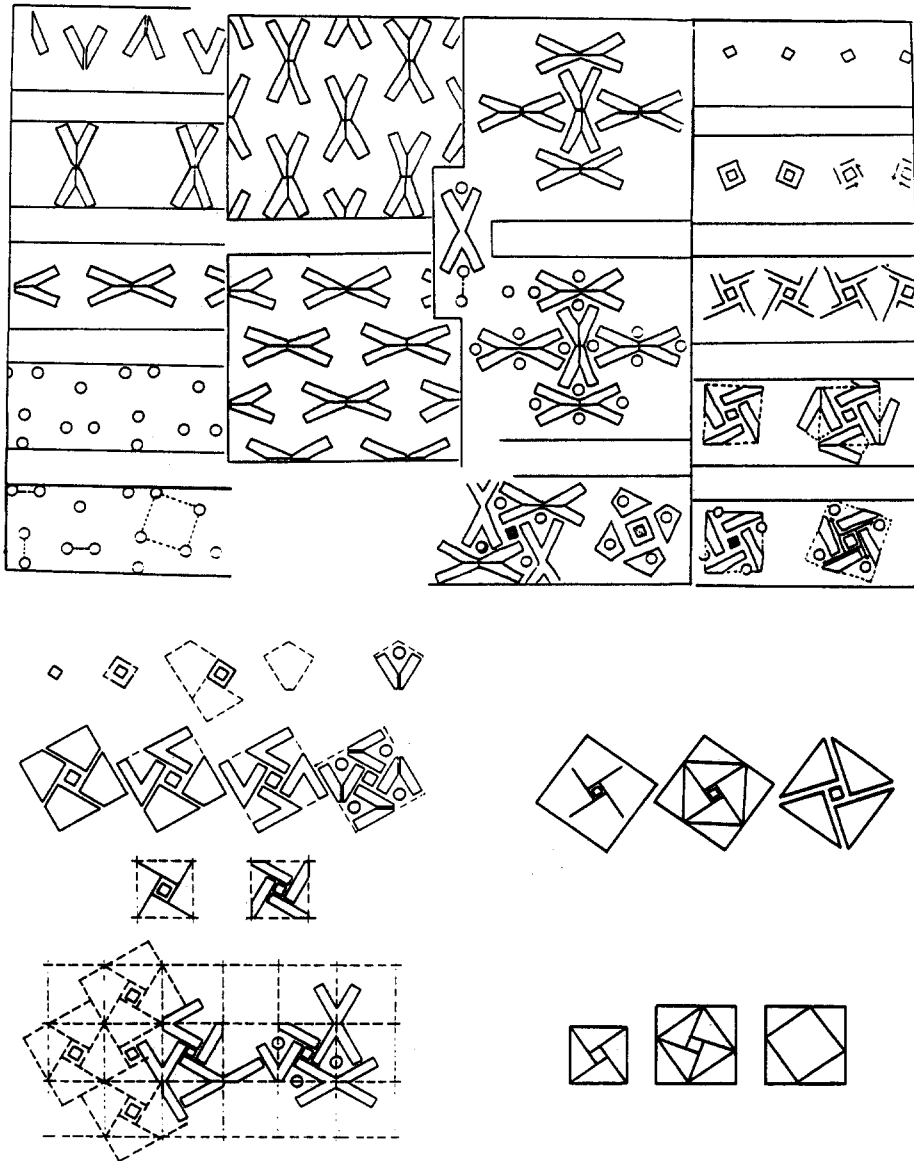


Fig. 6.3. Visual description and a sequential depiction of the underlying geometric structure of the Kharraqan brickwork pattern.

and I wondered what was being signified. This is the second influence, the fashion of linguistics, and its popularity was high at that time and persisted into the eighties. In the article "Symbols and signs in Islamic architecture", Oleg Grabar continues to use such terms of moneme, phoneme and semiotics: "a theme such as the muqarnas which involves nearly all morphemes of decoration . . ." [19]. What does that signify, I ask? If we are going to use linguistics, semiotics or appropriate another language, let us at least use one that can tell us something about the structure of this geometry. If we are going to keep picking up small elements, such as morphemes and larger elements such as monemes, what are we comprehending about the pattern or the design that is really there? There is also here a strong atomistic tendency that appears rooted in a predisposition by a certain political tendency among sociologists in particular and writers of contemporary Arab history to describe Islamic culture and civilization as "atomistic" and the society as "a mosaic". This maintained that there was no relationship between one part of that aggregate to the other. It was an aggregate at random. There was no structure. There were elements; everything was reduced to those elements. The art historians further legitimized this as a view by the way they looked at these Islamic

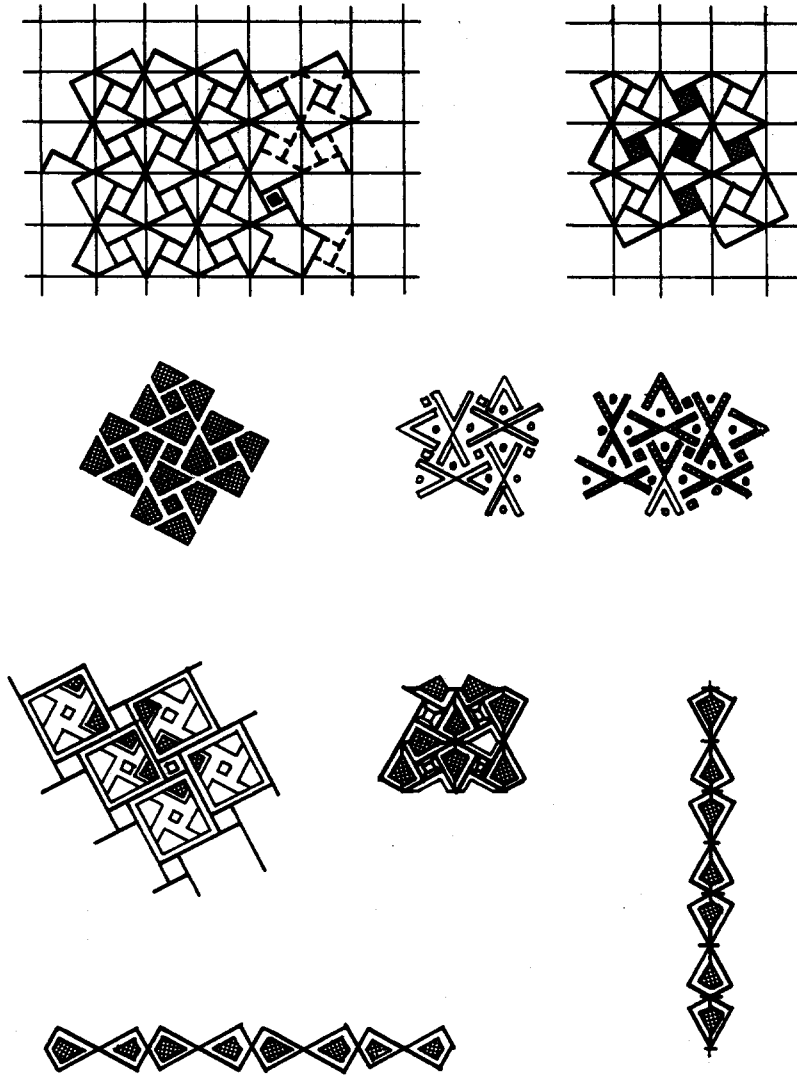


Fig. 6.4. Development of different patterns from the same underlying geometric structure of the Kharraqan brickwork pattern.

geometric patterns and mosaics. Yet if we continue to look at the pattern that we are describing, slowly and systematically, we cannot help seeing that there are some actual relationships that are taking place between these brick shapes; there is some expected order within the brick pattern; and it appears if we stop long enough to take a deeper look. Those geometric elements and brick shapes do not occur at random. “A whole is not the same as a simple juxtaposition of previously available elements”, Jean Piaget insists in his *Structuralism* [20]. For if we now look at the circles that form a square (Fig. 6.3), we will see in their midst a real square and around the square we will see four lozenges in a 4-fold rotational pattern. If we return to where we started, we must ask these questions: Where is the morpheme? Where is the moneme? What is being signified? Where is the semiotic relationship here? It is obviously clear that this kind of terminology, borrowed from the linguistic and semiotic languages, is not going to get us far; for it cannot even tell us that there is a strong geometrical structure underlying this visual pattern of bricks; nor can it tell us that this tapestry of brickwork has a meaning or symbolizes and signifies a specific concept. This is not to say that the sophisticated language and methodology of the science of Linguistics and Semiotics cannot be of use as an analytical tool in other fields, particularly in literary analysis; but in the case of geometric patterns, we have already a precise scientific language developed for this purpose.

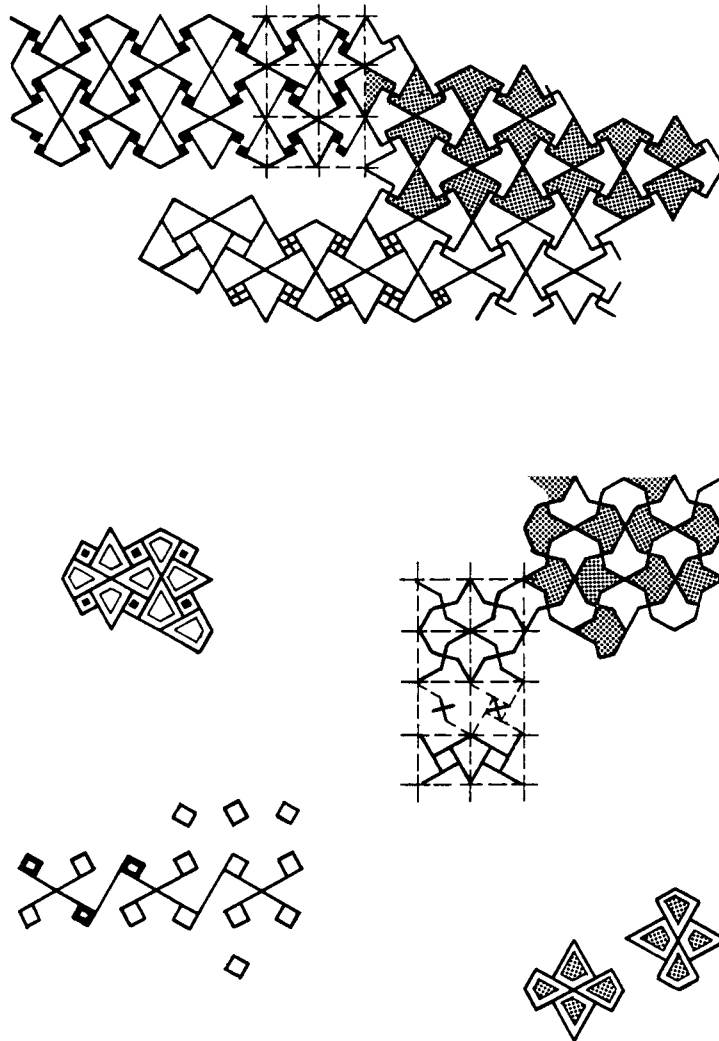


Fig. 6.5. Development of different patterns from the same underlying geometric structure of the Kharraqan brickwork pattern.

#### THEORETIC RELATIONSHIPS AND DESIGN STRUCTURE

Moreover, regarding the underlying structure of this pattern (Figs 6.2 and 6.3) and judging by the evidence found in the Paris manuscript “On interlocking similar and congruent figures” (*Fī tadākhul al-ashkāl al-mutashābiha aw al-mutawāfiqa*) that I had been working on (Fig. 7), four or five clear geometric construction steps can lead us to the underlying basic structure: if we use symmetry operations to operate on, or moved around the elements of the structure, then different relationships appear (Figs 6.4 and 6.5). In this way we have the development of different patterns from the same underlying elements. As J. Piaget had seen, there is a strong relationship among those elements that creates the process of composition, and together they create a whole. Compositions have laws that regulate them. By means of such reasoning, we come very close to group structure and Group Theory.

The final shape in the fifth step of the construction from the Paris manuscript (Fig. 7) is one of the most often used designs in Islamic art. It is most popular in woodwork and in ceramics. A very early example of its use in wood is in the door of the Mosque of Imam Ibrahim in Mosul (498 A.H./1104 A.D.) (Fig. 8.). It is used in ceramic in the side wall of the Iwan of Masjid-i-Jami in Isfahan, originally built around 515 A.H./1122 A.D. and redecorated later in 1112 A.H./1800 A.D. (Figs 9.1 and 9.2). This design is related to a very important problem in geometry, and has clear theoretical

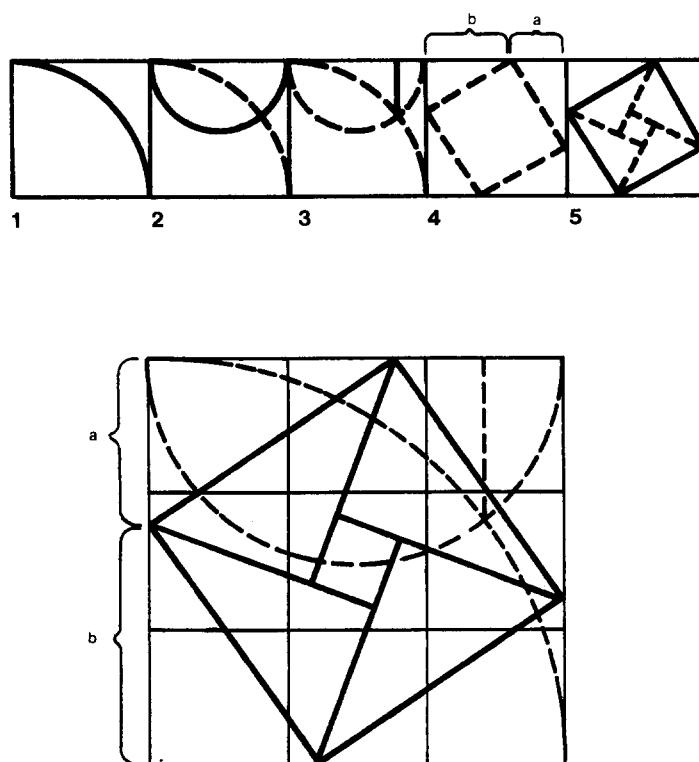


Fig. 7. Five-step construction of the underlying structure of the pattern as given in the Paris manuscript Ancien fond Persan No. 169.

or scientific ramifications. In the Islamic tradition of practical geometry, as early as the time of Abū'l Wafā' in the tenth century, there was a concentrated interest in the division of geometric areas such as a triangle and a square. The square gained specific attention in two respects: (1) how to divide it into a given number of squares (two, three or more), when the length of the side of one was known; and (2) how to construct a square equal in size to the sum of the areas of two, three or more given squares (Fig. 10). As we shall see, Classical Greek Geometry and the Pythagorean Theory deals with a specific case of these problems. The Greek method of proof for the Pythagorean Theory is given in Euclid's *Elements*, Book I, Proposition 47 (Fig. 11.1), which relies on a long proof of similar triangles and application of areas, while the method in the Islamic manuscripts is closer, in its dependence on dissection, to the Indian proof (Fig. 11.2) of Bhāskara (born 1114 A.D.), given by Sir Thomas Heath in his commentary on *Euclid's Elements* [21]. The Islamic approach for the artisans depends upon a practical method of proof (Fig. 12.1), in which the second square *b* is bisected into two equal rectangles; the rectangles are then cut through their diagonal into two triangles. The resulting four triangles are then placed around the square *c*, with their hypotenuses adjacent to or coinciding on the sides of the square *c*. In the middle a square area is left, the smallest square *a* is placed in it so as to fit the whole area. The visual clarity in this method allows the artisan to dispense with the need for logical proof of the relationship:  $a^2 + b^2 = c^2$ . This does not mean that the Muslim artisan and the scientist-geometrician, such as Abū'l Wafā', did not distinguish between the necessity for logical proof or for following methods of construction that the scientist had examined through correct proof, in contrast to the informal trial-and-error method. On the contrary, in his chapter "On the division of the squares and their combination", he stresses that the artisans should be aware that the reliance on trial and error in their constructions is not the route to correctness, even though the drawing might seem to appear as visually correct. On the contrary, the methods that the geometricians have demonstrated as correct through logical proof are the methods that artisans should follow because, upon repetition, these methods will always prove to be correct, unlike methods based on trial-and-error repetition or visual approximation. For a geometrician, once a problem is correctly proved, visual appearance is of no further importance,

even if the drawing appeared visually incorrect or correct. Abū'l Wafā' recounts that in an assembly of scientist-geometricians and artisans the two groups arrived at different methodologies for the construction of one square from the sum of three squares. The artisans wanted to use the method of dissection of the squares and adding the cut parts together to construct the larger square. They also brought several other methods, some of which could be proved, others not, although those that could not be proved correct still appeared visually correct to the eye or through the visual imagination of the viewer. He shows some of the incorrect usage, he says, in order to make the artisans aware of both the correct and the incorrect methods, and so that they would clearly know enough to not accept the incorrect methods. Ultimately, the clever and dextrous artisan would only depend upon the method of proof and not on that of trial and error [22].

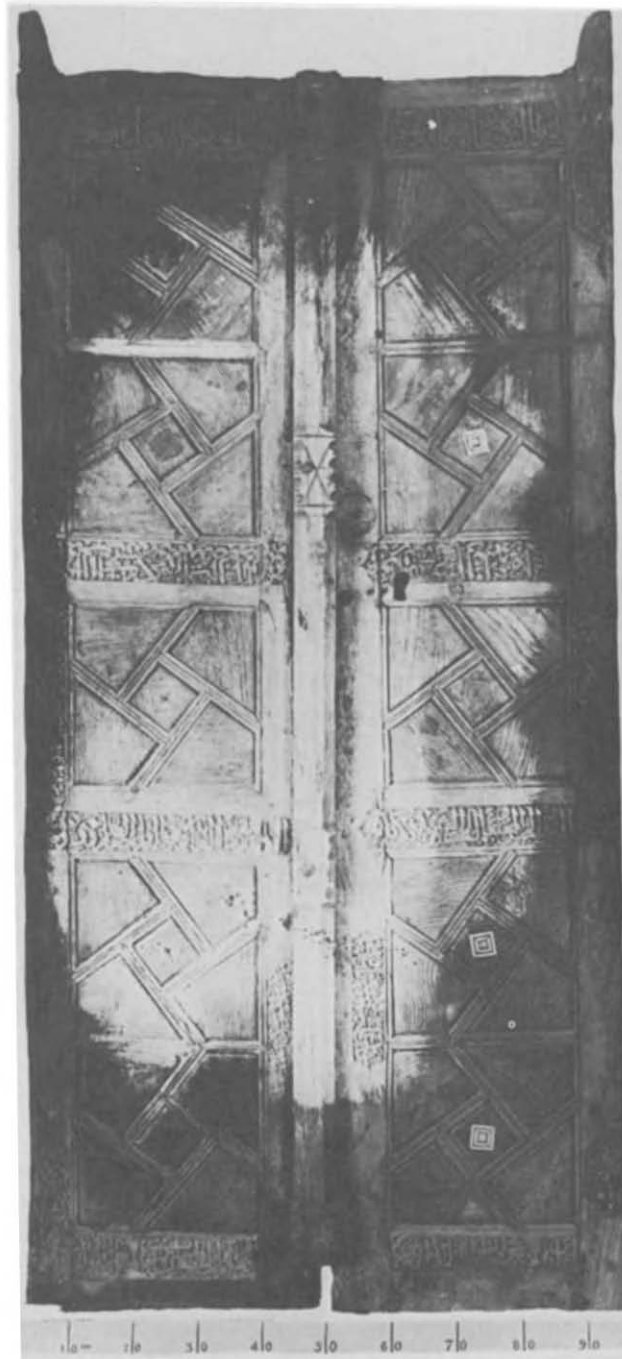


Fig. 8. Wooden door of the Mosque of Imam Ibrahim in Mosul dated 498 A.H./1104 A.D.



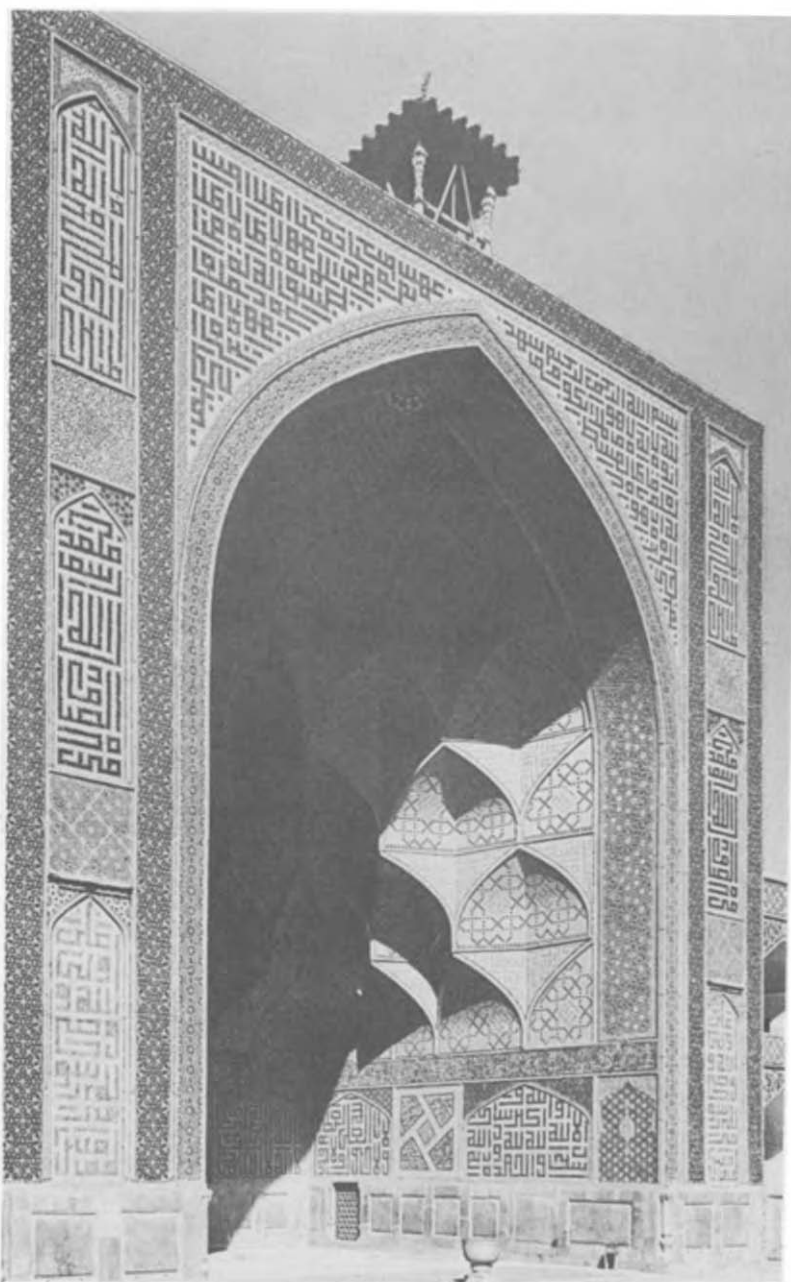


Fig. 9.1. The north-west Iwan of Masjid-i-Jami in Isfahan showing the use of the pattern in large scale in four places on the facade and inside the vault of the Iwan. Photo: S. P. and H. N. Seherr-Thoss, *Design and Color in Islamic Architecture*, p. 187. Smithsonian Institution, Washington, D.C. (1968).

This type of geometric algebra made available a number of mathematical and algebraic problems in drawn geometric illustrations that we see proliferating in these popular Islamic art designs, some of which are shown here (Figs 6.1–6.5, 8, 9.1 and 9.2), and which will appear again in the geometric problem from the Paris manuscript that will be discussed later (Figs 19.1–19.20). In rare instances, the architect–artisan seems to declare silently through visual evidence his precise knowledge of this geometric fact by placing his design in a prominent position of the decoration, as is the case with the treatment of the facade of the Iwan in the Isfahan, Masjid-i-Jami’ (Fig. 9.1), or by actually having the area at the heart of the central square of the design carry his name or signature “This is the work of Muhammad Ibin Mu’min Muhammad Amin . . .” (Fig. 9.2); obviously in this placement he is declaring to posterity that he “knows and knows that he knows . . .”, as the Arabic saying



Fig. 9.2. The ceramic panel showing the pattern in large scale and in the center of the small square carrying the name of the designer-artisan who made it. Photo: S. P. and H. N. Seherr-Thoss, *Design and Color in Islamic Architecture*, p. 189. Smithsonian Institution, Washington, D.C. (1968).

goes. The sides of the square (Fig. 12.2) indicate its division into two segments  $a$  and  $b$  where the sum of the side is equal to  $a + b$  and  $(a + b)^2 = a^2 + 2ab + b^2$ . Notice that in the case of the design of the Isfahan wall ceramic (Fig. 9.2), the size of length  $a$  is half of  $b$ , but does not necessarily have to be so. It is possible that this specific case of the ratios ( $a : b = 1 : 2$ ) is used by the artisan because it simplifies the task of measuring and cutting. Figures 12.1 and 12.2 follow this specific convention, while Fig. 7 shows the more general form of the theorem where  $a$  does not have a direct proportion to  $b$ , or where the side of the small inner square is  $b - a$ . Thus:

$$(b - a)^2 + 4\left(\frac{ab}{2}\right) = c^2,$$

$$(b - a)^2 + 2ab = c^2,$$

$$b^2 - 2ab + a^2 + 2ab = c^2$$

and

$$b^2 + a^2 = c^2 \quad \text{or} \quad a^2 + b^2 = c^2.$$

In these geometric designs, the Muslim artisan has demonstrated two different important relationships: the Pythagorean Theorem and the expansion of the second degree binomial.

Here it is appropriate to return to the mystical interpretation to discuss this specific design. K. Critchlow includes it in his book on *Islamic Patterns* (Fig. 13); he bases his analysis on the dodecagon and the square and remarks:

“This coincidence of the twelve and four suggests a zodiacal symbolism controlling or embracing the four axial kite shapes which can be taken to symbolize the four seasons, the four elements and the four qualities of heat and cold, moist and dry; the central sphere symbolizes the quintessence as a reflection of the bounding square.” [23]

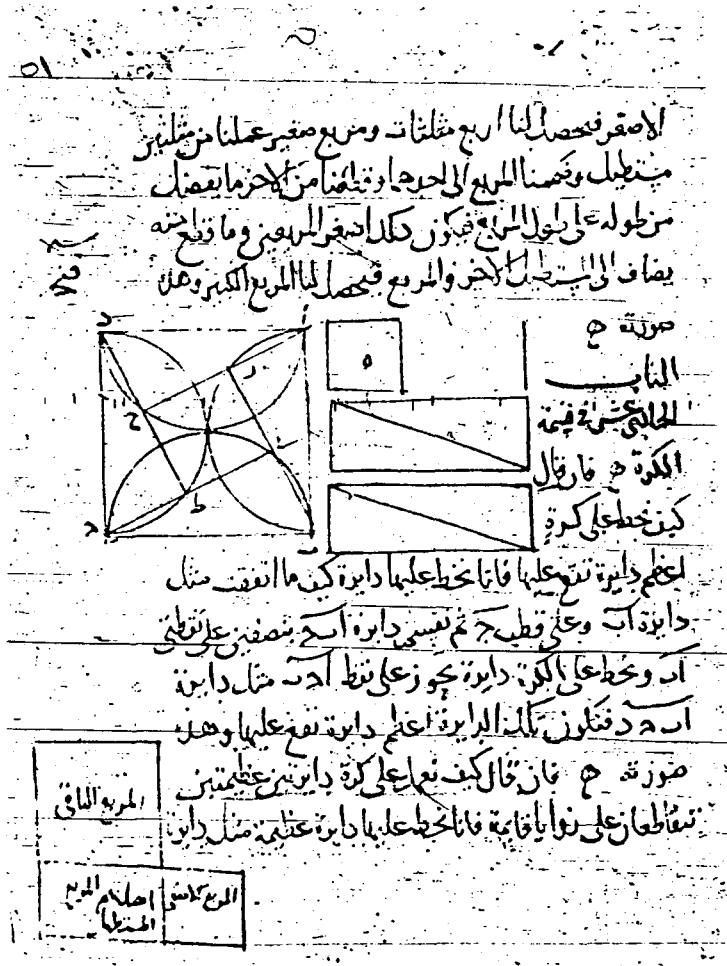


Fig. 10. Illustration showing one of the problems treating the construction of a large square. From the manuscript of Abū'l-Wafā' al-Būzjānī, Cairo, Dar al-Kutub.

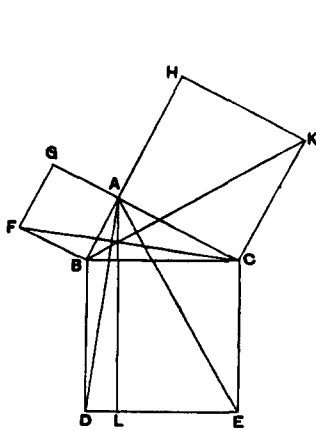


Fig. 11.1. The Greek method of proof of Euclid's Proposition 47. From Sir Thomas Heath, *Euclid's Elements* [21].

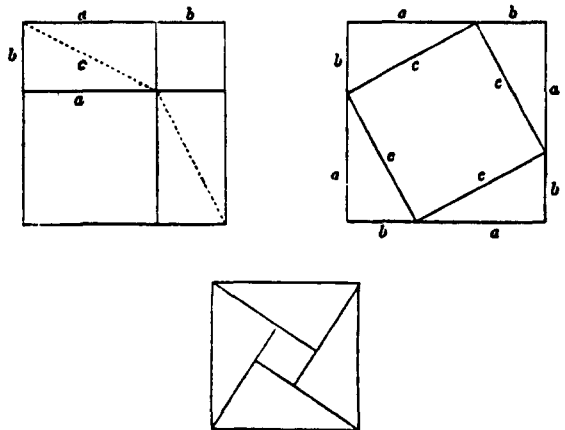


Fig. 11.2. The Indian method of Bhāskara. From Sir Thomas Heath, *Euclid's Elements* [21].

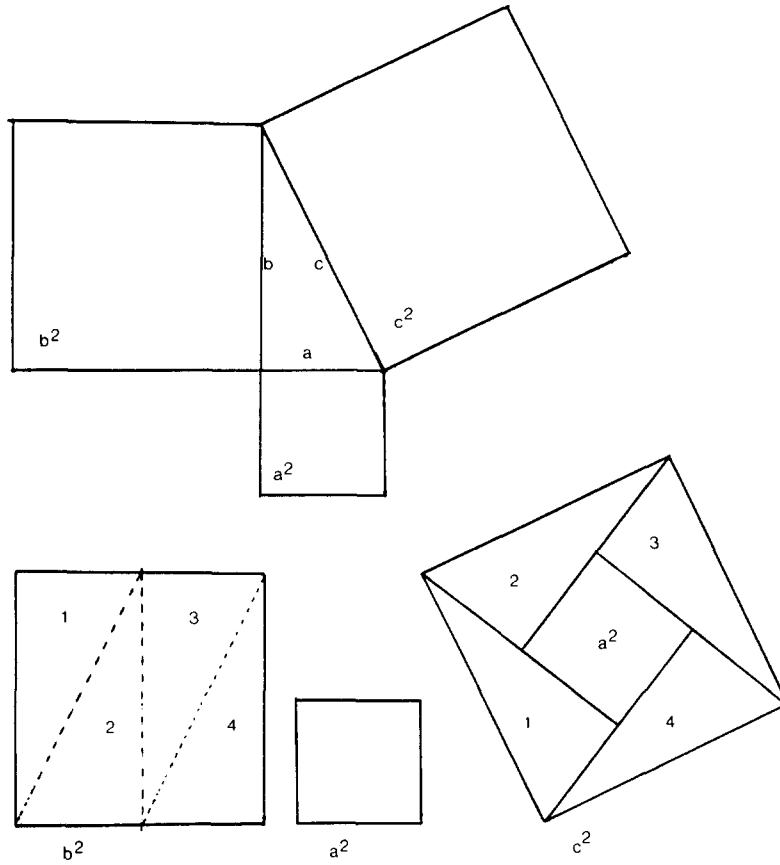


Fig. 12.1. The Islamic geometric design revealing the Islamic method for the visual presentation of the relationship  $a^2 + b^2 = c^2$ .

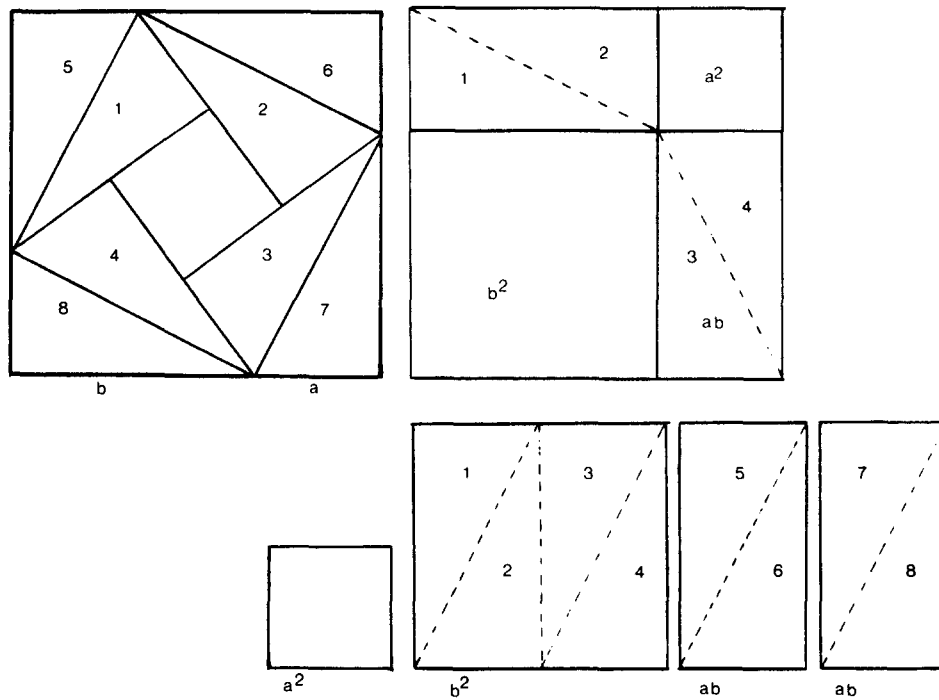


Fig. 12.2. The Islamic geometric design revealing the Islamic method for the visual presentation of the relationship  $(a + b)^2 = a^2 + 2ab + b^2$ .

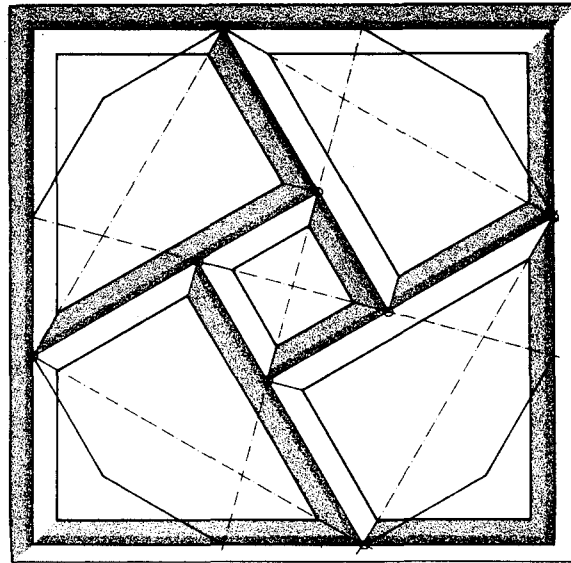


Fig. 13. The same Islamic design as given by K. Critchlow in *Islamic Patterns* [23].

I continue to question the attribution of these contemporary interpretations to the old traditional Islamic geometric forms. I wonder how it happened that, throughout the hundreds of folios of manuscripts I have examined dealing with Islamic geometric design I did not come across any such remarks or interpretation. Why did I not even find a marginal note, from later times, to this effect as we see in the illustration (Fig. 14) where there is plenty of space available for any one to subsequently add such a comment. As historians, our task is to come as close as possible to the original truth, depending on documents that are historically verifiable. The only comments in these manuscripts, that are non-scientific and that occur in these manuscripts, usually at the end of the given text and say: "... and Allah knows best". This is the only religious utterance that these Muslim scientists repeated time and again (Fig. 19; see Fig. 19.2), and there is no trace of any mystical or cosmological implication in this phrase. Rather, it reflects a very prominent Muslim belief in the humility of man before the knowledge of his Creator. Although the scientist is sure that his construction methods are correct because the geometric patterns produced are exact, still, even in this case of certainty, he humbly refrains from saying this is certain truth, but rather that true knowledge lies only with his God: Allah, the all knowing. This humble attitude of the scientist conforms to the general attitude or norm of Muslim beliefs.

#### THE NEED FOR A SCIENTIFIC LANGUAGE AND METHODOLOGY TO UNDERSTAND AND SYSTEMATICALLY CATEGORIZE AND DESCRIBE ISLAMIC GEOMETRIC PATTERNS

By the mid-seventies (1974–1976), I was fully concentrating on the Paris manuscript No. 169, "On the interlocking of similar or congruent figures", and working out in detail each problem found in it. In January 1977 something happened in my research. One day, the latest issue of *Scientific American*, January 1977, was brought to my attention because of an article in it on tiles and pattern that I could be interested in. As I looked at the cover, I recognized one of the geometric shapes that I was working on. This seemed unusual since the article was announcing contemporary scientific findings of a new mathematical relationship and a new geometric shape. I was sure, however, that I knew that geometric shape. Later, I took the magazine and some pages from my work and from the Islamic manuscript that I had been studying to two of my professors. It was a painful surprise that there was absolutely no recognition on their part of what I had seen nor of its significance. What was announced in *Scientific American*, by Martin Gardner in "Theory of tiles; extraordinary non-periodic tiling that enriches the theory of tiles", as a new finding by Roger Penrose of two shapes that can tessellate the space to infinity in a non-discrete group pattern (that extends to infinity

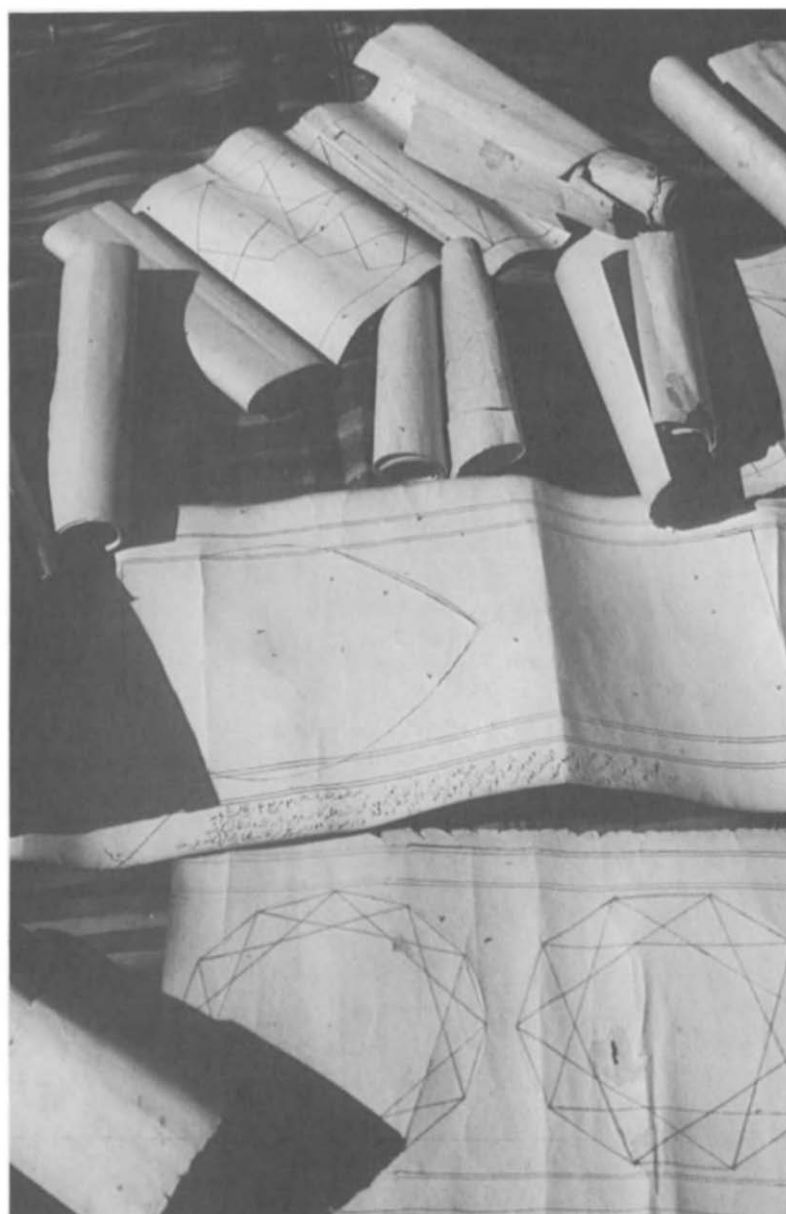


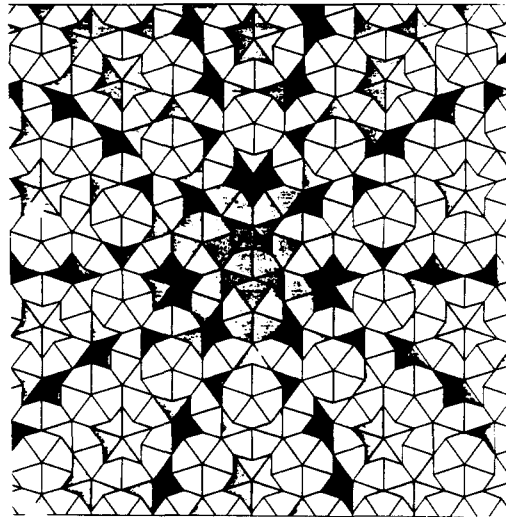
Fig. 14. Islamic artisan's scrolls with geometric drawings showing how marginal notes could be added.

but does not repeat) was not new! (Fig. 15) [24] Though the theory of non-periodic tiling does not appear, this geometric configuration of the shapes and pattern was in the Paris manuscript and it was described as a relationship derived from the decagon and a pentagonal star, named in the manuscript the “pentagonal seal” (Fig. 16). These so-called new shapes and pattern that are now even used by material scientists to analyze the structure of new materials (quasicrystals, shechtmanite) [25] and are now called Penrose tiling, have a geometric form or shape that has been known for hundreds of years by Muslim scientist–designers. Only the method of drawing it and the relationships it involves are different: while the modern Penrose method depends on the golden ratios and golden triangles, stressing the value of the angles, the old Islamic method uses the center angle for the decagon but stresses the ratio and relationship of the length of the radii of the decagon and its sides, also stressing the proportions of the lengths of these lines. This will be discussed in more detail in a future publication.†

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†The details of this problem will be published in a joint paper with Arthur Loeb in the near future.

# SCIENTIFIC AMERICAN



THEORY OF TILES

\$1.50

January 1977

Fig. 15. The cover of *Scientific American*, with decagon and pentagonal star of the Penrose tiles [24].

At this time I was still plowing through chemistry books, and one day by chance I came across one whose title attracted me: *Color and Symmetry* by Arthur Loeb, who taught at Harvard [26]. I attended his first class meeting of the second semester of the academic year. I showed him the material I was working on and the article by Gardner which he not only knew but had discussed in his last class. The wisdom in an Arabic saying declares:

“He who knows not and knows not that he knows not, shun him. And he who knows not and knows that he knows not, awaken him. And he who knows and knows that he knows, follow him.”

Quickly I found that A. Loeb had the language I was seeking, and that he had clarified and made more accessible to art students the complicated language of various symmetry notation systems and even the International Tables of Crystallography (Fig. 17). My study of symmetry had led to many books. Most pertinent to the arts was Hermann Weyl’s classic, *Symmetry* (1952), HSM. Coxeter’s text, *Introduction to Geometry* (1961) and A. V. Shubnikov and V. A. Koptsik’s *Symmetry in Science and Art* (1972; translated into English by G. D. Archard, 1974) [27]. None of these were designed to serve the specific needs of art and design. Rather, they expanded on the discussion of symmetry; the second used the notation of the International Tables of Crystallography, and the third included immensely detailed and exhaustive enumerations that go far beyond what an art historian needs.

A good example of the multiplicity of languages and the confusion of tongues is the comparison table showing seven of the more popular notation systems for the Plane Symmetry Groups (Fig. 18), included within a very short article, “The Plane Symmetry Group: their recognition and notation” by Doris Schattschneider in *The American Mathematical Monthly* (June–July 1978) [28]. In contrast to these, A. Loeb’s notation system clarified each and every center point of the symmetry group; gave the symmetry value at each of the symmetry roto-centers (which is of great use to the art student and to the designer artist or architect); explained the role of the interaction of circles in space and revealed the properties of space, fundamental regions and unit cells, reflections and enantiomorphism, and distinguished between mirrors and glides by underlining the symmetry value number for the centers that fall on mirror lines and using an inverted  $v^{(\wedge)}$  over the symmetry number of glide point centers. Moreover, his system could be easily taught to students of art, in a very short time,

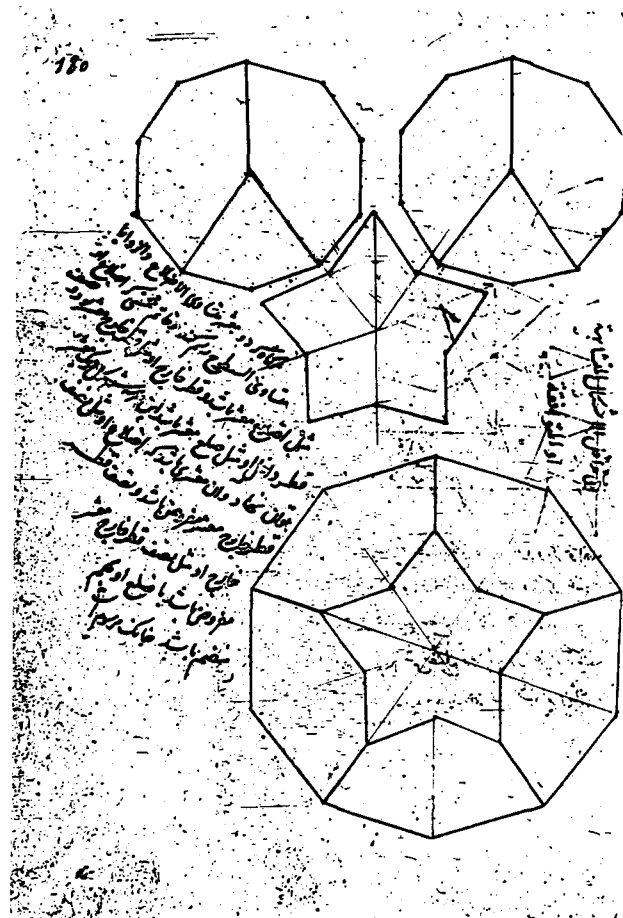


Fig. 16. Folio 180a from the Paris Islamic geometric manuscript "On the interlocking of similar or congruent figures", showing the design with the decagon and pentagonal star similar to the Penrose tiles.

for in a sense a large part of it is geared to them. Most essential is that he also recognizes that the problem we face in the field is again one of a multitude of languages, that this is causing a confusion of tongues, and that, ultimately, we have to choose one language in which to communicate with each other.

The multiplicity of different notational systems has also already crept into the computer world. By 1987 at least half a dozen software graphics programs based on symmetry became available on the market. Each used its own notational system and codes or code names to the symmetry patterns of Group Theory that the programs produced. In some cases, an approximate word is used that describes an image, such as flower, snow flake, or another fancy word far less precise than the original language developed by the crystallographers in 1935. The books on symmetry do not do any better and they tend to add to the confusion caused by a multiplicity of tongues. The academic community concerned with this subject must decide how to re-establish the use of a consistent and specific set of notation as a standard reference for this limited number of patterns.

The importance of Group Theory and its notational system for Islamic art lies in the fact that it provides a tool for exact cataloging of the infinite number of geometric designs used in Islamic art. It is also helpful as an analytical tool in recognizing the symmetry used within a design. Moreover, it provides a precise language and terminology by which those who are interested in these patterns can communicate precisely with each other about these patterns. All this might seem redundant to the scientists who have been involved in the study of symmetry yet, for the art historians, it is still an unacknowledged tool.



Table 3 Exhaustive list of configurations of symmetry elements in the plane

Number of distinct rotocenters with finite symmetry value	Combination of symmetry values	Configurations	I.T. notation	Illustrated by Figure number	
0	1	1		32	
		1m			
	∞	∞			8
		∞mm'			36
		∞m			35
		∞g			34
	1∞∞	1∞∞'		p1	20
		1∞∞'mm'		pm	37
		1∞∞'mg		cm	39
		1∞∞'gg'		pg	38
1	k	k	C <sub>k</sub>	7b	
		km	D <sub>k</sub>	33	
2	22∞	22'∞		12b	
		22''∞		41	
		22'''∞		40	
3	236	236	p6	15, 18	
		236	p6m	42	
	244	244'	p4	17	
		244''	p4m	44	
		244'''	p4g	43	
	333	33'3''	p3	16	
33'3'''		p3m1	46		
333'		p31m	45		
4	2222	22'2''2'''	p2	23	
		22'2''2''''	pmm	52	
		222'2''	cmm	51	
		222'2''g/g'	pgg	48	
		222'2''m/g	pmg	49	

Fig. 17. The Plane Symmetry Group notation table of A. Loeb [26].

COMPARISON OF NOTATION FOR THE PLANE SYMMETRY GROUPS

Internat'l (short)	Pólya; Guggenheimer	Niggli	Speiser	Fejes Tóth; Cadwell	Shubnikov-Koptsik	Wells Bell & Fletcher
p1	C <sub>1</sub>	C <sub>1</sub> <sup>I</sup>	C <sub>1</sub> , Abb. 17	W <sub>1</sub>	(b/a)1	1
p2	C <sub>2</sub>	C <sub>2</sub> <sup>I</sup>	C <sub>2</sub> , Abb. 18	W <sub>2</sub>	(b/a):2	2
pm	D <sub>1</sub> kk	C <sub>4</sub> <sup>I</sup>	C <sub>2</sub> <sup>I</sup> , Abb. 19	W <sub>1</sub> <sup>2</sup>	(b/a):m	3
pg	D <sub>1</sub> gg	C <sub>4</sub> <sup>II</sup>	C <sub>2</sub> <sup>II</sup> , Abb. 20	W <sub>1</sub> <sup>3</sup>	(b/a):b̄	4
cm	D <sub>1</sub> kg	C <sub>4</sub> <sup>III</sup>	C <sub>2</sub> <sup>III</sup> , Abb. 21	W <sub>1</sub> <sup>1</sup>	(a/a)/m	8
pmm	D <sub>2</sub> kkkk	C <sub>20</sub> <sup>I</sup>	C <sub>20</sub> <sup>I</sup> , Abb. 22	W <sub>2</sub> <sup>2</sup>	(b/a):2·m	5
pmg	D <sub>2</sub> kkgg	C <sub>20</sub> <sup>III</sup>	C <sub>20</sub> <sup>III</sup> , Abb. 24	W <sub>2</sub> <sup>3</sup>	(b/a):m·ā	6
pgg	D <sub>2</sub> gggg	C <sub>20</sub> <sup>II</sup>	C <sub>20</sub> <sup>II</sup> , Abb. 23	W <sub>2</sub> <sup>4</sup>	(b/a):b̄·ā	7
cmm	D <sub>2</sub> kgkg	C <sub>20</sub> <sup>IV</sup>	C <sub>20</sub> <sup>IV</sup> , Abb. 25	W <sub>2</sub> <sup>1</sup>	(a/a):2·m	9
p4	C <sub>4</sub>	C <sub>4</sub> <sup>I</sup>	C <sub>4</sub> , Abb. 26	W <sub>4</sub>	(a/a):4	10
p4m	D <sub>2</sub> <sup>+</sup>	C <sub>40</sub> <sup>I</sup>	C <sub>40</sub> <sup>I</sup> , Abb. 27	W <sub>4</sub> <sup>1</sup>	(a/a):4·m	11
p4g	D <sub>2</sub> <sup>-</sup>	C <sub>40</sub> <sup>II</sup>	C <sub>40</sub> <sup>II</sup> , Abb. 28	W <sub>4</sub> <sup>2</sup>	(a/a):4⊙ā	12
p3	C <sub>3</sub>	C <sub>3</sub> <sup>I</sup>	C <sub>3</sub> , Abb. 29	W <sub>3</sub>	(a/a):3	13
p3m1	D <sub>3</sub> <sup>+</sup>	C <sub>30</sub> <sup>I</sup>	C <sub>30</sub> <sup>I</sup> , Abb. 31	W <sub>3</sub> <sup>1</sup>	(a/a):m·3	15
p31m	D <sub>3</sub> <sup>-</sup>	C <sub>30</sub> <sup>II</sup>	C <sub>30</sub> <sup>II</sup> , Abb. 30	W <sub>3</sub> <sup>2</sup>	(a/a)·m·3	14
p6	C <sub>6</sub>	C <sub>6</sub> <sup>I</sup>	C <sub>6</sub> , Abb. 32	W <sub>6</sub>	(a/a):6	16
p6m	D <sub>6</sub>	C <sub>60</sub> <sup>I</sup>	C <sub>60</sub> <sup>I</sup> , Abb. 33	W <sub>6</sub> <sup>1</sup>	(a/a):m·6	17

CHART 6. Sources referred to in the table are listed in the References. The groups are listed in consecutive order as they appear in the International Tables of X-ray Crystallography, [13]. Note that Speiser interchanges the Niggli notations of C<sub>30</sub><sup>I</sup> and C<sub>30</sub><sup>II</sup> (figure numbers in the Speiser column are for the 2nd, 3rd, and 4th editions of his book).

Fig. 18. D. Schattschneider's comparative table of notation for Plane Symmetry Groups [28].

## AN EXAMPLE OF THE PROCESS OF ISLAMIC GEOMETRIC DESIGN

Stepping beyond the symmetries of Islamic geometric art will lead us to the mechanism of the process of Islamic geometric design. It is only when we follow the step-by-step procedure of constructing the geometric design that we come to understand it in full. The manuscript evidence and documentation thus becomes of primary importance, for it alone can lead us to the heart of this matter. It is not enough to give a quick translation or even an edition of the text in the original languages: neither can by itself lead to comprehension of the process of design. This is why it is essential to study in detail the scientific manuscripts and old documents that are now available to us, for they can bring us closest to the true historical procedure of the Islamic science of design. It is also essential to understand their scientific significance, because only through this can we place them in the larger context and recognize their importance in the science of geometric design. For instance, the following example from the Paris manuscript folio 192b turns out to be a fascinating one in its use of a strict algorithm with irrational numbers. Though based on a very strict algorithm and set of proportion, as we shall see, this geometric method of design and its forms is not a closed, dead-end system. It is rather one whose strength lies in its simplicity and strictness of derivation, for these two characteristics give it the open power for generating an infinite number of design variations from a single simple set of proportions.

In terms of working with primary sources, folio 192b is a problematic one since it has four illustrations and only three texts (Fig. 19). One assumes that three illustrations have a text while one illustration does not, and that the text closest to each illustration belongs to it. After translating the texts and comparing it to the illustrations, I found that text and illustration did not seem to correspond. I therefore proceeded to analyze the geometry of the illustration and reconstruct the design. Time and again, I came back to the texts and the folio that had now become exactly like

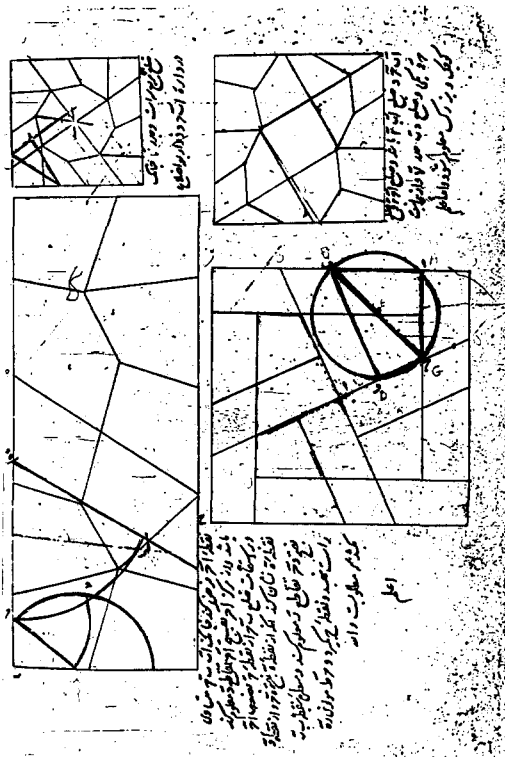


Fig. 19. A sample geometric problem from the Paris manuscript "On the interlocking of similar or congruent figures", folio 192b.

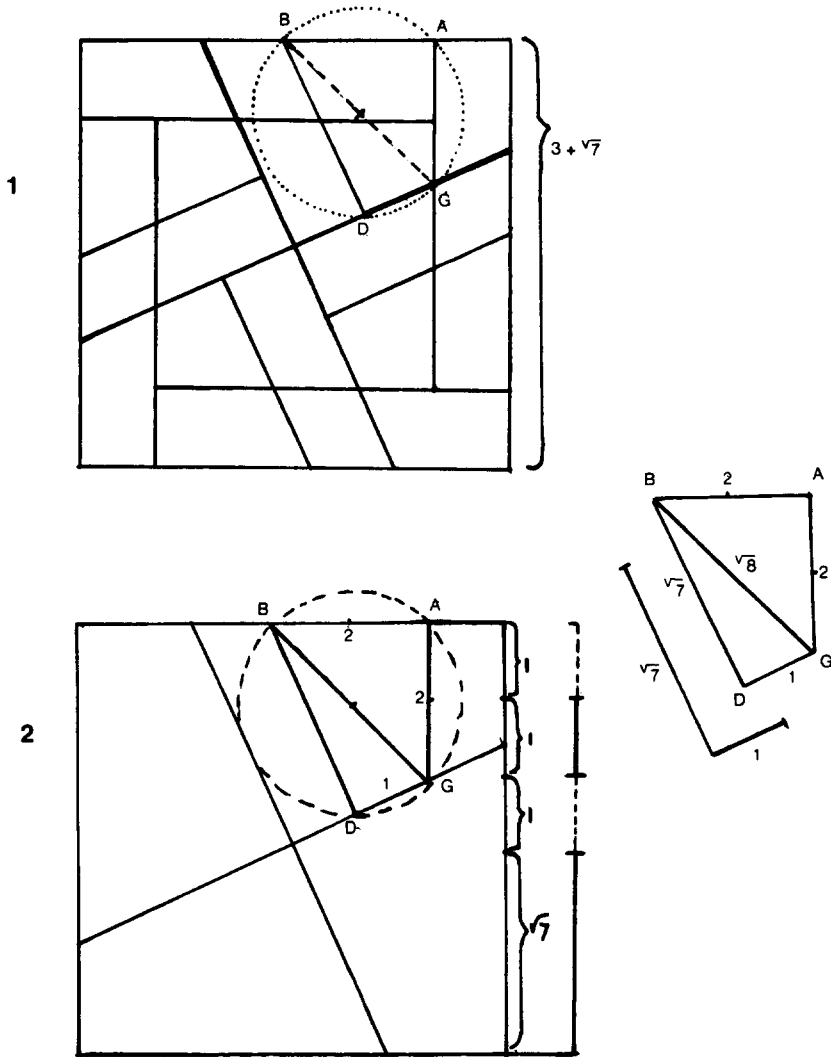


Fig. 19.1. The given geometric problem designed with a square unit:  $3 + \sqrt{7}$ .

Fig. 19.2. The given quadrangle ABGD with proportions  $1:2:2:\sqrt{7}$  inscribed with a circle.

a puzzle. While I was analyzing the geometry of the illustration in the upper right corner, some specific numbers and irrational numbers emerged. One day as I began reading the text in the upper left corner, the numbers looked familiar. Suddenly, I realized that these numbers related in values to the asymmetric quadrangle of the illustration in the upper right side of the folio. The pieces of the puzzle fell together. The two upper texts should be linked to make one text, and this one text belonged not to the illustrations next to which it was copied, but to the illustration in the middle right side of the folio. Clearly this was a copyist's error, which in turn tells us something else about this unique manuscript: that it is copied and thus there must have been another manuscript like it. My main hope here is that the original manuscript has survived somewhere and will be recognized or discovered one day. Using another manuscript, one can resolve some of these textual problems.

The art historian Midhat S. Bulatov dealt with this problem in his book *Geometric Harmonization in the Architecture of Central Asia of the 9th–15th Centuries* (*Geometricheskaiia Garmonizatsiia Arkhitektury Srednei Azii IX–XV vv.*), published in Moscow in 1978 (the text of the original Persian manuscript was translated into Russian by Vil'danova). I differ with M. S. Bulatov on several points of reconstruction regarding this problem which he treats very briefly. He improperly concludes that this problem was not accompanied by an explanatory text. He states that "the following description of the construction (in the original text of the manuscript) does not correspond to the drawing,

which is decoded in the following way . . .” [29]. In other words, the problem is solved independently from the text of the manuscript even though Vil’danova had translated the first two texts that belong to this geometric problem (illustrated in the middle right side of the folio).

When dealing with original documents, it is permissible to take some liberty in proposing the solutions to the problems if there is no text available for the drawn geometric illustrations. However, in these solutions would only be approximations to the process of the original Islamic method. After all, there could be as many different ways of handling a problem as there are contemporary persons interested in treating it. One should treat these documents with the same precision and respect which any historical document is given. Every detail of the original has to be worked out in full, to attempt to approximate as closely as possible the original truth, just as any historian would document an episode in the past. After one has verified it, then there are interpretations and implications that could be suggested or given.

In the case of treating geometric drawings or illustrations in manuscripts one has to examine the original physical manuscript. This is a vital issue, for many construction marks were only done by the sharp needle end of the copyist’s metal divider to lightly scratch the paper surface marking these points of construction that were not inked. This fact means that photographs and microfilms are inevitably incomplete documents for they cannot depict these uninked marks and the original must be checked for any complete investigation. In general, the Soviet teams of scholars are limited to work on microfilms of manuscripts and without being able to examine the original manuscripts. They should be commended for the efforts that they have invested in returning to these primary sources which the rest of the Western scholars have demonstrated a skeptical attitude toward, a lack of interest in, as well as an incapability of retrieving and handling them. Second, one would wish that they had more access to some of the main issues and specific scientific findings on the international scene that are of great aid in recognizing the global significance of these original documents in the history of Islamic sciences.

The following segment gives the steps of the process of this design and their implications as they relate to other issues of interest or concern to geometry and design:

*Fig. 19.1.* In the texts the length of the side of the square is given as:  $3 + \sqrt{7}$ .

*Fig. 19.2.* The quadrangle **ABGD** inscribed inside the circle has as the length of its sides: side **AB** = 2 units, side **AG** = 2 units and side **DB** =  $\sqrt{7}$ . And the following comment is made: “And from here the small and large proportions are determined, and Allāh knows best.”

Let us look in detail at the mathematical specifics of this quadrangle and see how it is generated in a very simple but strict algorithm.

*Fig. 19.3.* This shows an isosceles right-angle triangle whose sides are each given the length of 2 units, making its hypotenuse **BG** =  $\sqrt{8}$ .

*Fig. 19.4.* The midpoint of the hypotenuse **BG** is taken as the center of a circle and an arc of this circle is drawn such that angle **A** is on its circumference, such that all three vertices of the triangle lie on the circumference. The length of 1 unit compass opening is marked as **D** from **G** on the circumference, where **GD** = 1. Point **D** is connected to **B** by line **DB**. Line **DB** =  $\sqrt{7}$ , it follows that angle **D** is a right angle.

*Fig. 19.5.* This shows the second right-angle triangle with proportions  $1:\sqrt{7}:\sqrt{8}$ .

*Fig. 19.6.* These two triangles together make the kite shape **ABGD**.

*Fig. 19.7.* All vertices of the kite-shaped asymmetric polygon **ABGD**, with side lengths of 1, 2, 2,  $\sqrt{7}$ , lie on the circumference of the circle.

*Fig. 19.8.* When mirror-reflected on the side **BD** of length  $\sqrt{7}$ , the quadrangle kite-shape creates a semiregular pentagon that has all of its sides equal (2 units), while its two opposite angles **A** and **A'** are both right angles.

*Fig. 19.9.* This shows how a cruciform area of a width equal to 1 unit is created when a gnomon with the width = 1 added from points **A** (Fig. 19.9b) and **D** (Fig. 19.9c), on both sides **AG** and **BD**, moving **A** to **A'**, **B** to **B'**, **G** to **G'** and **D** to **D'**. A third strip unit with width equal to 1 unit is measured from point **A** on **AG** and drawn parallel to **AB** (Fig. 19.9d). Adding the first two gnomons of 1 unit to the quadrangle **ABGD** makes it retain its original proportions  $1:2:2:\sqrt{7}$  in the larger asymmetric quadrangle kite shape **A'B'C'D'** (Fig. 19.9e), which is then rotated four times at its right angles **D'** to form the large square unit (Fig. 19.9f).

*Fig. 19.10.* This shows the side equal to 1 unit measured three times, and  $\sqrt{7}$  adding to  $3 + \sqrt{7}$ .

The 1-unit strip (as a result of the third strip seen in Fig. 19.9) is shown running around the borders of the square.

Fig. 19.11. The quadrangle is rotated within the square, showing all the added lines as a result of the addition of the gnomon; also, the different segments of proportions  $1:2:\sqrt{7}$  are depicted on the side, as the rotated quadrangle generates the larger square unit of side length  $3 + \sqrt{7}$ .

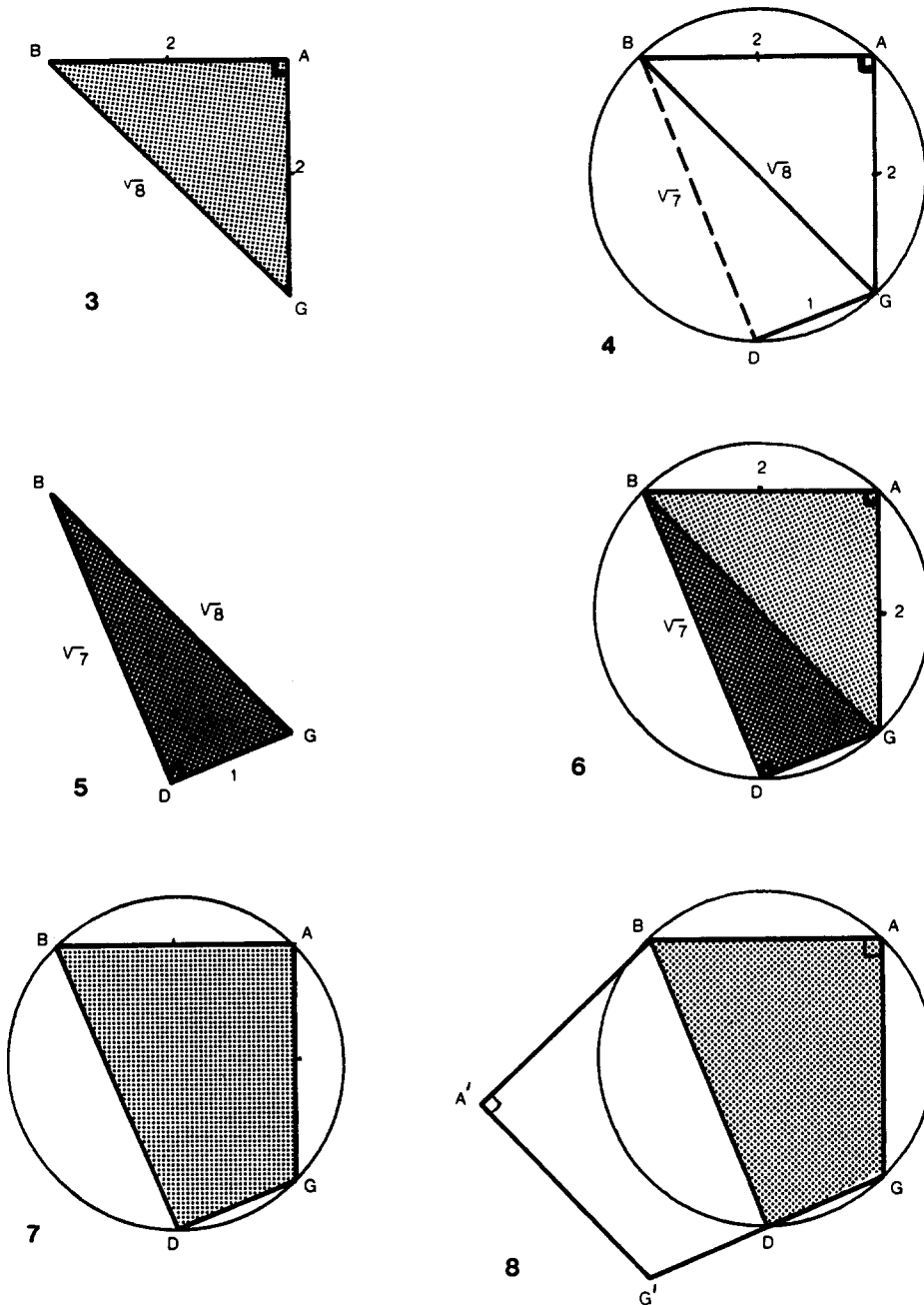


Fig. 19.3. The first triangle is a right-angled isosceles with proportions  $2:2:\sqrt{8}$ .

Fig. 19.4. The first triangle is circumscribed and a unit of length 1 is marked on the circumference.

Fig. 19.5. The second triangle  $DBG$ , proportions  $1:\sqrt{7}:\sqrt{8}$ .

Fig. 19.6. The two triangles combined to form the asymmetric quadrangle  $ABGD$ .

Fig. 19.7. The asymmetric quadrangle  $ABGD$  with sides  $1, 2, 2, \sqrt{7}$ .

Fig. 19.8. The semiregular pentagon with side lengths of 2 units.

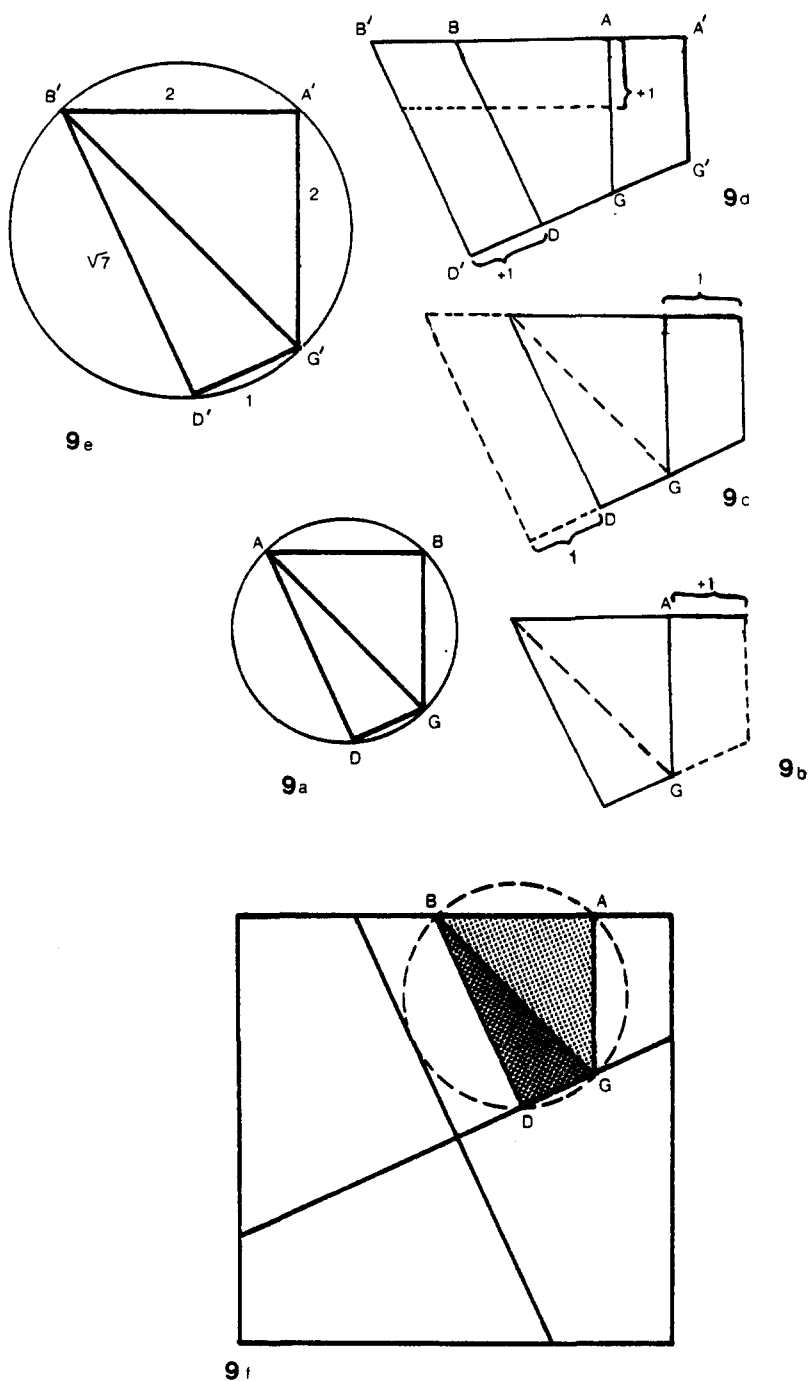


Fig. 19.9. The addition of gnomon strips to enlarge the quadrangle but retain the original proportions.

Fig. 19.12. The final drawing of the 4-fold rotation around a point is seen. This is made possible due to the characteristic proportions and opposite right angles of the symmetric quadrangle **AGBD**.

Fig. 19.13. For easier visual reading, the quadrangles are colored. The sides of the square indicate its division into two segments (a) and (b) where: the sum of the side is  $= a + b$ ; and  $(a + b)^2 = a^2 + 2ab + b^2$ .

Fig. 19.14. When we apply the symmetry operations of mirror reflection and glides to the whole unit a pattern will develop to tessellate the plane, in a  $244$  or  $p4g$ . It has a 4-fold roto-center through which two perpendicular glide lines pass, to carry them to the next enantiomorphic 4-fold symmetry center. The 2-fold roto-centers are located at the corners of the square, lying on two mirrors inter-

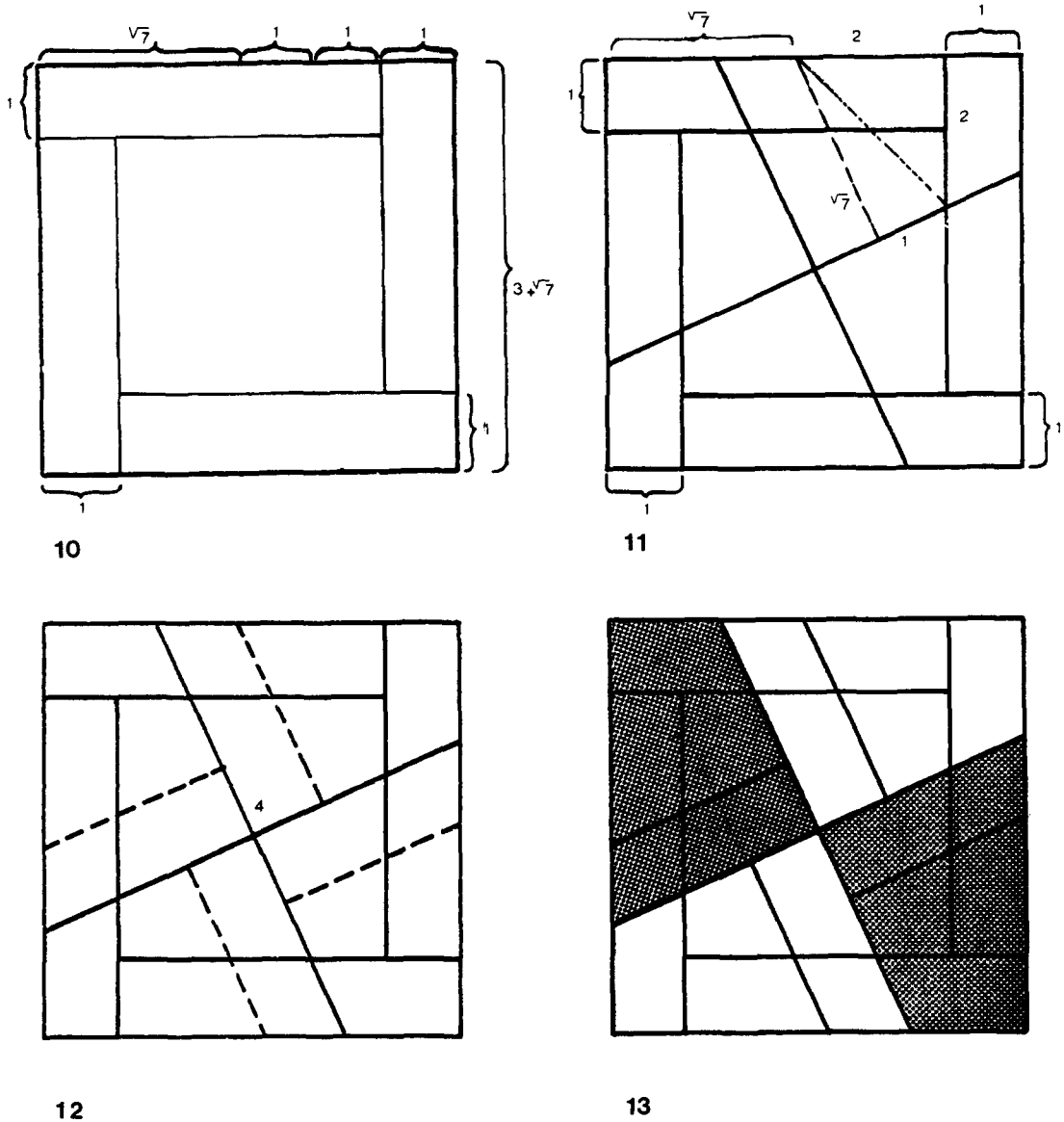


Fig. 19.10. The large square unit of side length  $3 + \sqrt{7}$ .  
 Fig. 19.11. The different proportions of the quadrangle in 4-fold rotation to generate the large square unit of side length  $3 + \sqrt{7}$ .  
 Fig. 19.12. Center for the 4-fold rotation of the quadrangle  $ABGD$  with all the line subdivisions.  
 Fig. 19.13. The quadrangles shaded for easier visual identification, dividing the side into two segments.

secting at right angles. So the 2-fold centers lie on mirrors, one 4-fold rotational symmetry center is a glide image of the other 4-fold rotational symmetry center.

*Fig. 19.15.* This shows the pattern colored in the simplest manner to reveal four small kite shapes rotating around the center of each square. The coloring also reveals how this pattern would be a very good subject for ceramic or woodwork, requiring only three different shapes: a small symmetric kite, a lozenge and a quadrangle.

*Fig. 19.16a.* When the quadrangle is taken without the subdivisions and is repeated through the symmetry operations, we can clearly see the semiregular pentagons and how they tessellate the plane in 4-fold symmetry  $244$ . The two opposite right angles of the semiregular pentagon allow for the 4-fold rotation.

$$\underline{2}44^{\wedge} = P4g$$

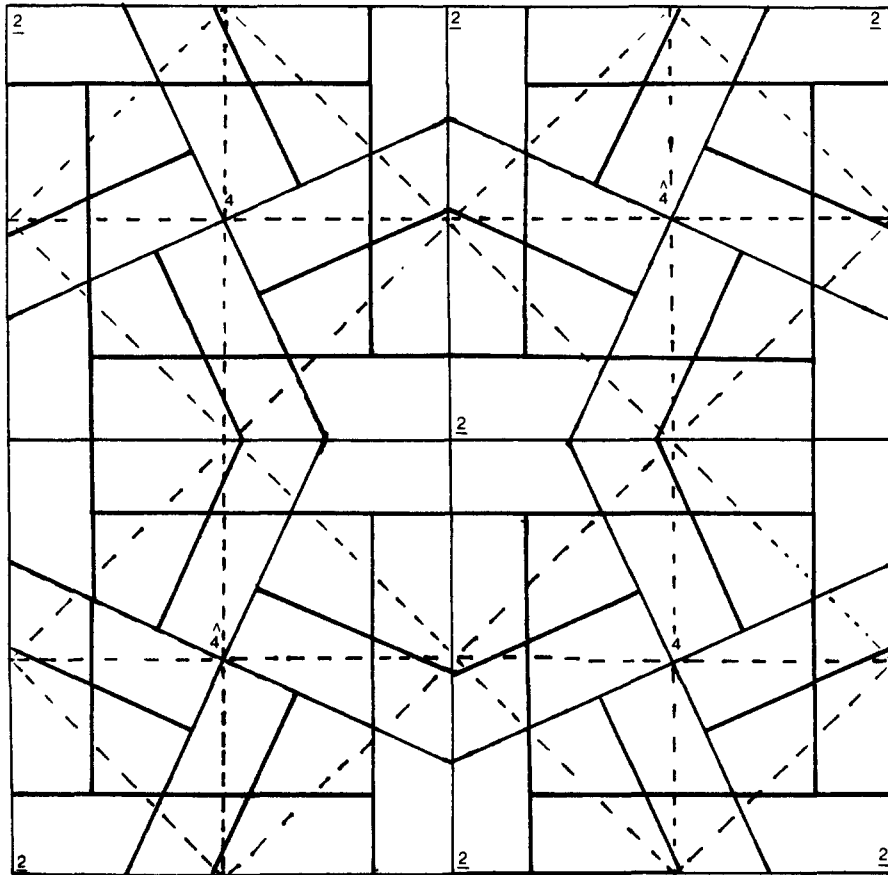


Fig. 19.14. The quadrangle repeated in a  $244^{\wedge}$  symmetry operation; the 2-fold centers are on perpendicular intersecting mirror lines, while the two 4-fold centers are on glide lines.

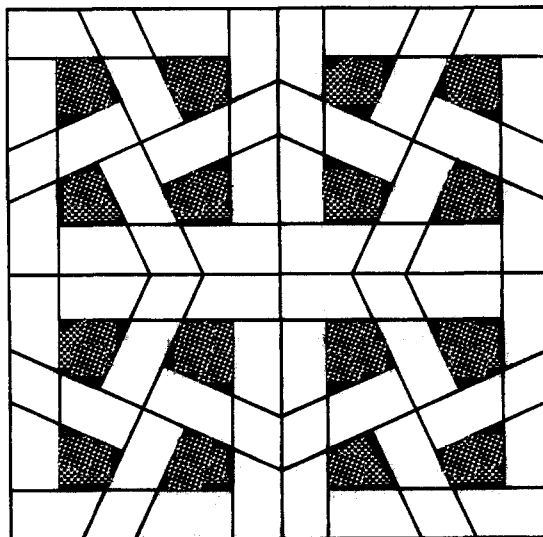


Fig. 19.15. A simple shading for the pattern.



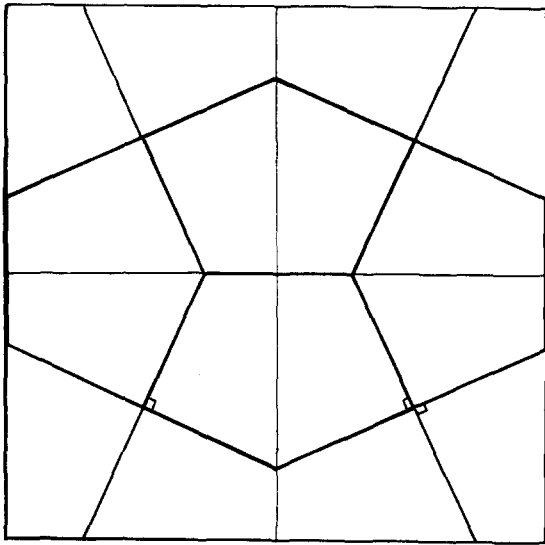


Fig. 19.16a. The repeated quadrangle without the subdivisions creates 244 tessellation with semiregular pentagons.

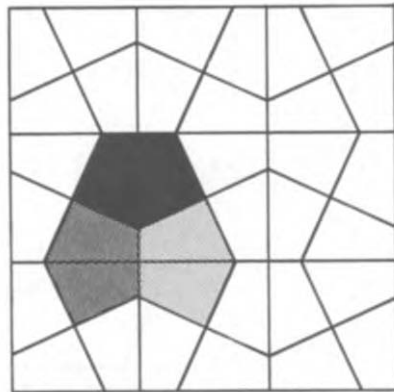


Fig. 19.16b. The pentagons are shaded for easier visual identification.

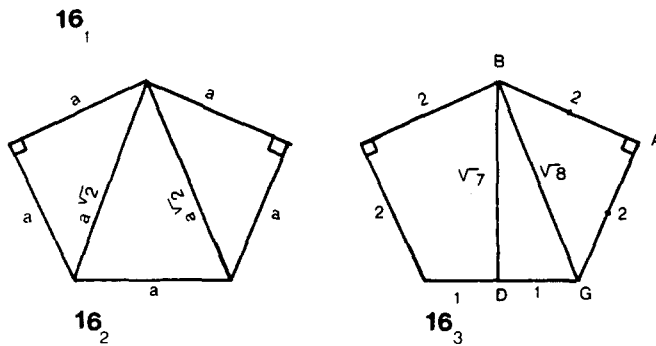
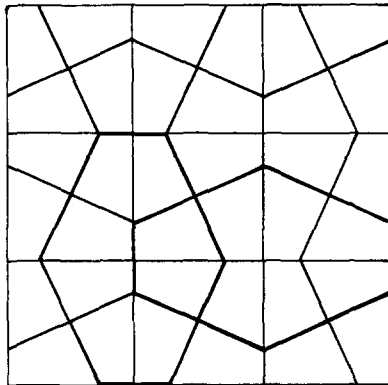


Fig. 19.16c<sub>1-3</sub>. The pattern of a “favorite street tiling in Cairo” of a tessellation with pentagons and a comparison of the Islamic and Western derivation of the semiregular pentagons.

Fig. 19.16b. Here, the semiregular pentagons are colored in three shades to make them easier to see.

Fig. 19.16c<sub>2-3</sub>. Two different semiregular pentagons are drawn at the bottom of the page. On the right side is the Islamic pentagon, where  $\sqrt{7}$  is the critical value in the design. On the left is the Western one given by J. A. Dunn in an article on “Tessellations with pentagons” [30]. Dunn’s

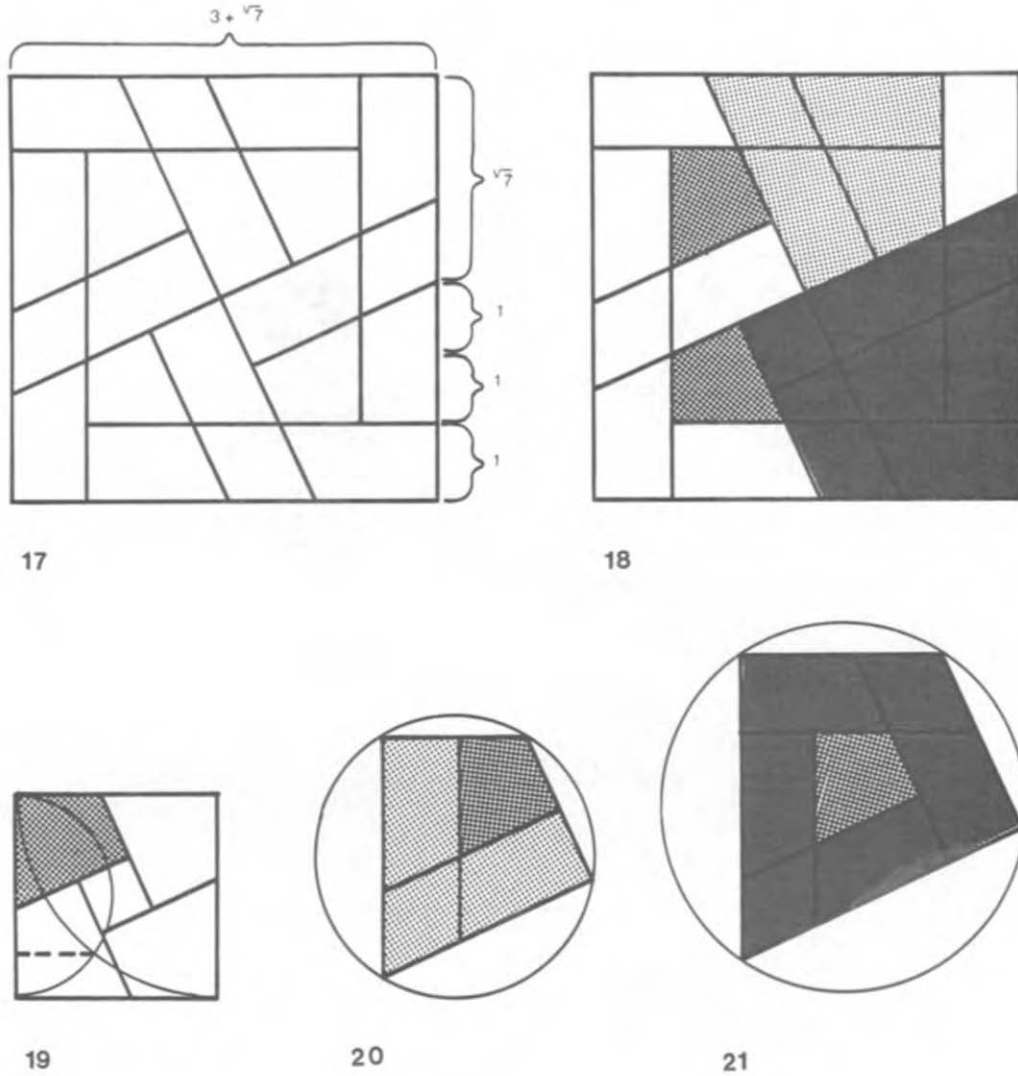


Fig. 19.17. The 4-fold rotation of the asymmetric kite-shaped quadrangle  $ABGD$  leaves no empty space at the center of the larger square unit  $(3 + \sqrt{7})$ .

Fig. 19.18. The pattern is shaded to show three similar but different sized symmetric kite shapes.

Fig. 19.19. The small symmetric kite shapes in 4-fold rotation around a point leave a square at the center.

Fig. 19.20. The symmetric kite shape in the medium size.

Fig. 19.21. The symmetric kite shape in the larger size.

pentagon has an isosceles pentagon triangle that has the critical length of  $a\sqrt{2}$  for the two equal sides while the third side is a or any given length.

This tiling (Fig. 19.16c<sub>1</sub>) is referred to as the “favorite street tiling in Cairo”. In it, the tessellation is considered hexagonal, each hexagon being a combination of four semiregular pentagons. However, this tessellation is based on the 4-fold rotation of the semiregular pentagon, with sides equal to two units and two opposite right angles. The latter characteristic permits the 4-fold rotation of symmetry group  $244$  or  $p4g$ .

Fig. 19.17. The addition of the gnomons of 1-unit width on sides  $AG$  and  $BD$  of the quadrangle kite shape of side proportions  $2:2:1:\sqrt{7}$  allows it to retain the original set of proportions in the larger form, as was seen in Fig. 19.9. When this larger asymmetric kite-shaped quadrangle is rotated four times around a point, it creates a larger square (the square of sides  $3 + \sqrt{7}$  given in the text of the manuscript) leaving no empty area at the center.

*Fig. 19.18–19–21.* Within the square we see that there are three symmetric kite-shaped quadrangles of different sizes that have mirror symmetry, making each of the two sides equal to the other (note Figs 19.9 and 19.11). The kite shapes are similar in proportions but have three different sizes.

*Fig. 19.19.* When these symmetric kite shapes are rotated four times around a point, they form a larger square and leave a smaller square at the center. We can now see how this pattern relates to the design on the wooden door of Imam Ibrahim, Mosul, and to the design from the Isfahan Masjid-i-Jami (Figs 7–8). The sides of the square indicate its division into segments *a* and *b* where the sum of the sides is  $= a + b$ , and  $(a + b)^2 = a^2 + 2ab + b^2$ . This is exactly like the first set that had the theoretical concentration, presented in the visual description and the structural analysis from the brick wall of the Kharragan Towers.

The main lesson that can be learned from this detailed example of documented manuscript evidence of the process of Islamic geometric design is that there is a specific way to design and construct each geometric design. Clearly, one can generalize that each problem of geometric design that is documented has its own steps of construction and that other than the Diophantine equation [31],

$$\frac{1}{k} + \frac{1}{l} + \frac{1}{m} = 1,$$

contrary to the claims of the adherents of the mystical interpretation in the seventies, there is no other single formula for deriving all the geometric patterns in Islamic art. The so-called unique method of construction is based upon the subdivision of the circle; it is not a replacement of a scientific formula and it cannot be propagated as the unique way to derive and construct all Islamic geometric patterns. The science of symmetry of patterns tell us that there are 17 different periodic two-dimensional groups and 7 groups periodic in a single direction (string or ribbon), also that each of these groups could have an infinite number of different designs. As seen, these Islamic geometric manuscripts give us samples of the infinite design variations of the basic 17 periodic groups; these documented geometric problems or examples in turn could be the basis for developing many new sets of design.

This last group of illustrated designs was an exercise in which I tried to explore the generative and creative power that, I believe, is at our fingertips once we understand this rich tradition of Islamic geometric design. This special quadrangle of side proportions  $1:2:2:\sqrt{7}$  is carried through various symmetry operations to produce over 80 designs. A small sample of them are presented here to show the design potential of this geometric form:

*Fig. 20.1.* A 2-fold symmetry operation rotating it 180° on each side in  $22'2''2'''$  point groups that generates a pattern of these asymmetric kites that tessellate the plane.

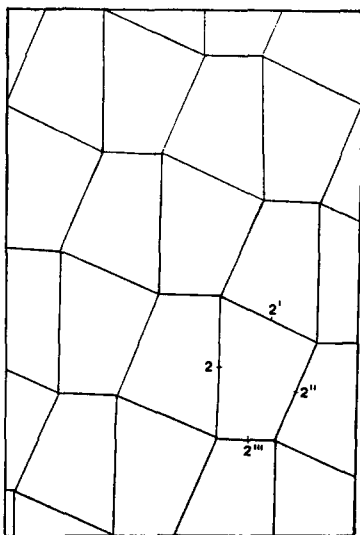


Fig. 20.1. The asymmetric quadrangle ABGD in a  $22'2''2'''$  symmetry pattern.

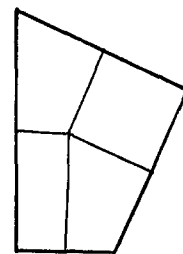


Fig. 20.2. The asymmetric quadrangle ABGD subdivided with perpendicular bisectors.

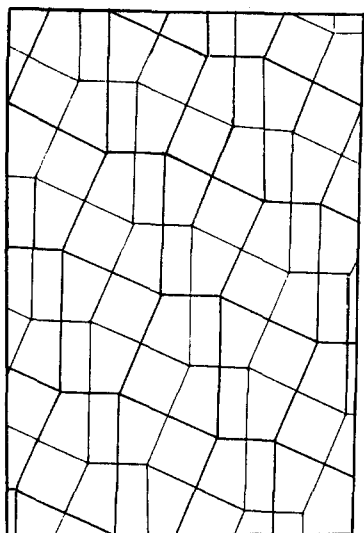


Fig. 20.3. A 22'2''2''' tessellation with the subdivided quadrangle ABGD.

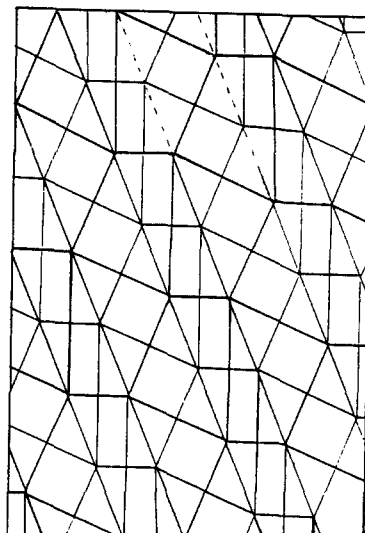


Fig. 20.4. The diagonal of the smaller quadrangles are drawn.

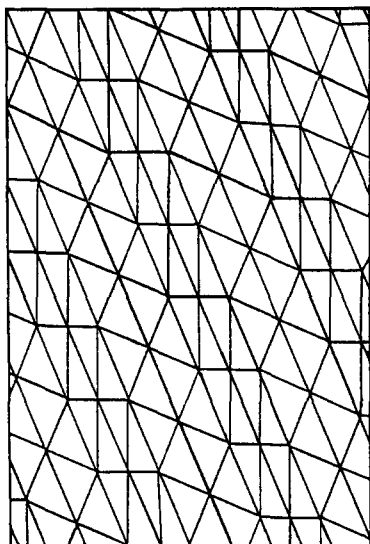


Fig. 20.5. The diagonals of the squares and rectangles are drawn.

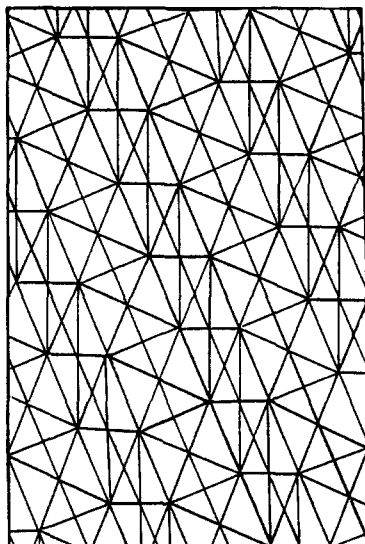


Fig. 20.6. The second diagonals for the square and rectangles are drawn.

*Fig. 20.2.* When the perpendicular bisectors of all the sides are drawn, the quadrangle is divided into four areas, generating four small quadrangles:

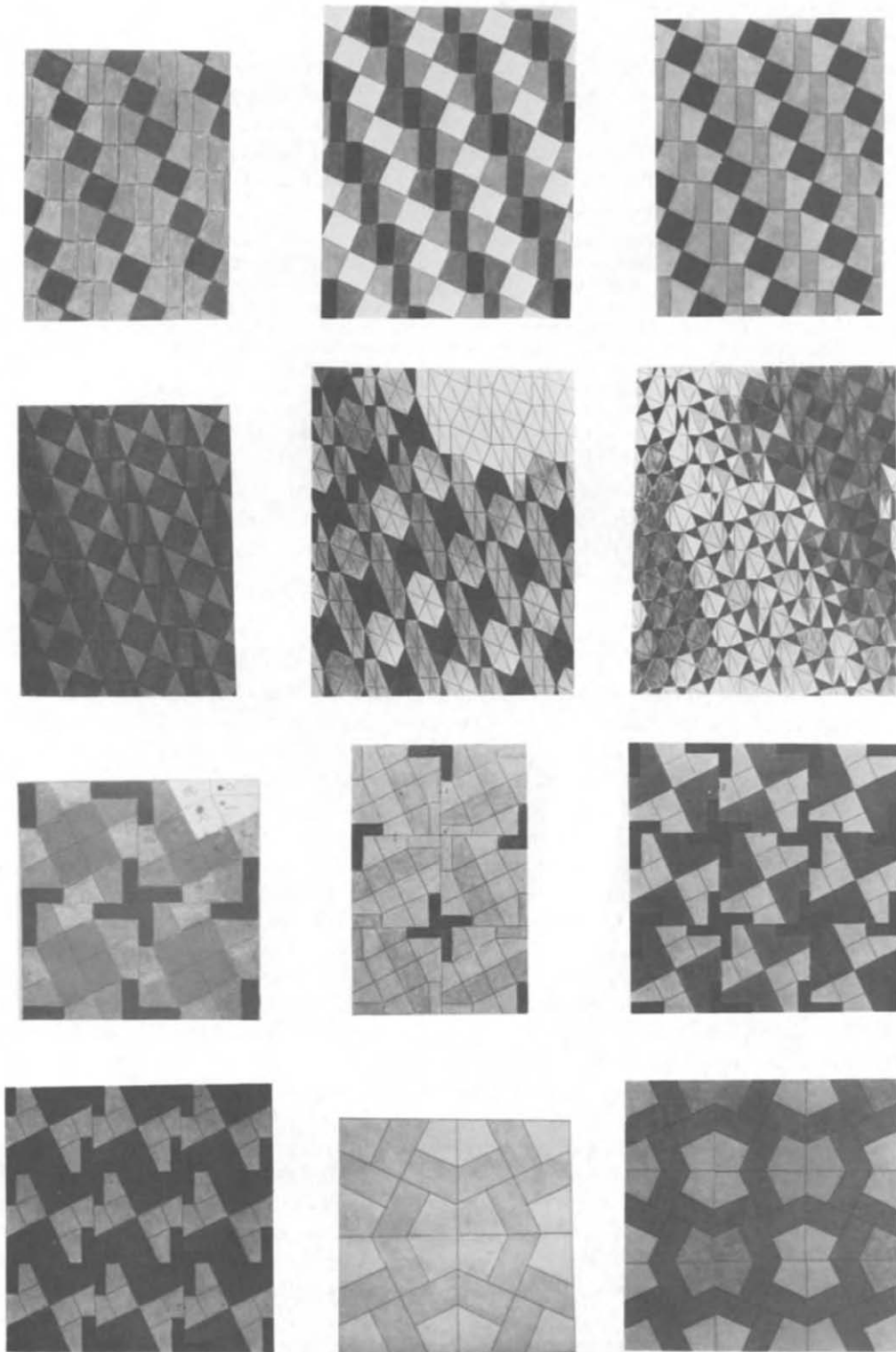
1. A small square of a 1-unit side length.
2. A rectangle with side lengths equal to 1 and 1/2.
3. A small asymmetric kite-shaped quadrangle.
4. A small asymmetric kite-shaped quadrangle.

The two small quadrangles, that are the result of the subdivision, are both similar to each other and to the original quadrangle, retaining the original side proportions  $1:2:2:\sqrt{7}$ .

*Fig. 20.3.* The subdivided quadrangle is used to tessellate the plane in a 22'2''2''' pattern.

*Fig. 20.4.* The subdivision is increased by drawing diagonal lines for the smaller quadrangle.

*Fig. 20.5.* The subdivision is increased by drawing the diagonal lines for the smaller square unit and the smaller rectangle.



Figs 21.1.–21.12. Colored examples (unfortunately reproduced in black and white) of patterns generated from the asymmetric quadrangle  $ABGD$ .

*Fig. 20.6.* The second diagonals are drawn for the smaller square and smaller rectangle.

*Figs 21.1–21.12.* The patterns resulting from 2-fold and 4-fold rotations of the asymmetric kite-shaped quadrangle  $ABGD$  with proportions  $1:2:2:\sqrt{7}$  generate a number of patterns. These patterns in turn are multiplied through further subdivisions and selective symmetric coloring of the

shapes to develop an infinite number of patterns, some examples of which are seen in these photographs.

I chose this special quadrangle because I felt that its geometry was very strict and its algorithm unique. In fact it is so strict and yet somehow so simple that, at first glance, the asymmetric quadrangle or kite shape it produces looks boring or not visually interesting. However, when used through different symmetry operations and coloring it showed the potential to generate an infinite number of patterns. In this regard, I would like to quote A. Loeb from his article "Algorithms, structures and models":

"We have observed that our perception of an apparently complex configuration is altered when, instead of attempting a complete description of the object, we generate the configuration from a small number of relatively simple modules together with an algorithm for assembling them. . . .

"Generally, we do not know the modules and algorithms which would generate a given complex configuration. The role and process of science would seem to consist of a search for appropriate modules and algorithms which generate models whose behavior resembles adequately that of the complex configuration studied. The analogy between model and observed configuration is limited and quite subjective, depending on the observer, and the purpose, context and background of the experiment.

"In design, the algorithmic approach generates with simple means a rich repertoire of patterns transcending the repertoire of the 'naked eye'. In addition the conceptual component of such a generated pattern has an esthetic appeal of its own, and constitutes an important link between art and science." [32]

If to A. Loeb "the role of science is to search for appropriate modules and algorithms which generate models whose behavior resembles" the simple algorithm and shape we have seen in this Islamic design, then art too has to search for the proper scientific languages and tools to generate new forms and expressions.

## CONCLUSION

Over 10 years ago, I stood firm against attempts to direct me away from the core contents of the Islamic manuscripts of practical geometry in order to deal with the issues of Princely and Royal patronage, the circles of the intelligentsia, or the intellectual activities of tenth century Baghdad and its role in the writing of the first manual of practical geometry that we have. At that time I asked these questions: Where does the key to the existence and creation of these texts lie? Is it in the specific achievements and readiness of science at that time and in that place? Is it in the patronage of the royal intellect? Is it in the interests and practicality of the age? Is it in the person of a particular scientist-geometrician and his interest in and play with the scientific materials that were available to him? Or is it in the intrinsic needs of the arts, artisans and architects, which in turn ultimately directed scientists to create these texts. Is that not their true *raison d'être*?

The true patron of the scientists who wrote these ancient manuscripts was art. It was the artisans and the architects who called for the services of science and scientists to assist them in solving the design problems that they were facing. And as in the case of Islamic art in the past, science must come to the service of the arts, whether we are talking today of Islamic art, of Western art or of art generally, today more than ever before, for otherwise I cannot imagine how the arts can move into the twenty-first century.

In the mid-seventies people completely shunned attempts to return to Islamic tradition. Today, turning toward the tradition is, for some, the fashionable thing to do and a fashionable subject to focus international conferences on. For others, it is something frightful to be avoided, for it implies a reactionary conservative return to medieval times, and is feared as being the source of militant radicalism. Throughout Islamic history, there has always been a voice declaring that the Muslim tradition is relevant to contemporary times. I have tried, visually, to show here that returning to the study of medieval Islamic tradition does not necessarily mean to advocate a move from the present century backwards to medieval times, whether in the field of science and geometric design as in the case here, or in other fields. Rather, the opposite may be the case, for Islamic tradition is so strong that, if we are in touch with the language of the present time and ground ourselves in this strong old tradition, we can arrive at an expression that is not only contemporary but that could be meaningful and valid in the coming century. (The design examples presented here are already more than 10 years old.)

"And it was revealed unto Muhammad, peace be upon Him to 'Say: Are those who know equal with those who know not?' But only men of understanding will pay heed." [33]

*Acknowledgements*—This paper is based on two public lectures: “In the Tower of Babel: history, history of science, scientific theory and the process of Islamic geometric design”, presented at MIT on 3 and 7 November 1987. A personal note of appreciation should be added here for the support given by Dr Arthur Loeb and Dr Hazem Sayed for the generous support they gave in the making of the lectures and this article.

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